STRENGTH OF MATERIALS

SLOCUM-HANCOCK

Date Due

17 1945			
JUN 4 1951			
FEB 20 1960			
~~FEB 11 1962~~			
~~NOV 21 1968~~			
MAR 25 1962			
DEC 20 1974			
~~DEC 23 1977~~			
~~OCT 1980~~			
~~1992~~			
~~FEB 28 1994~~			

MATHEMATICAL TEXTS

Edited by PERCEY F. SMITH, PH.D
Professor of Mathematics in the Sheffield Scientific School of Yale University

---

Elements of the Differential and Integral Calculus (Revised Edition)
By W. A. GRANVILLE, PH.D.

Elements of Analytic Geometry
By P. F. SMITH and A. S. GALE, PH.D.

New Analytic Geometry
By P. F. SMITH and A. S. GALE, PH.D.

Introduction to Analytic Geometry
By P. F. SMITH and A. S. GALE, PH.D.

Advanced Algebra
By H. E. HAWKES, PH.D.

Text-Book on the Strength of Materials (Revised Edition)
By S. E. SLOCUM, PH.D., and E. L. HANCOCK, M.SC.

Problems in the Strength of Materials
By WILLIAM KENT SHEPARD, PH.D.

Plane and Spherical Trigonometry and Four-Place Tables of Logarithms
By W. A. GRANVILLE, PH.D.

Plane and Spherical Trigonometry
By W. A. GRANVILLE, PH.D.

Plane Trigonometry and Four-Place Tables of Logarithms
By W. A. GRANVILLE, PH.D.

Four-Place Tables of Logarithms
By W. A. GRANVILLE, PH.D.

Theoretical Mechanics
By P. F. SMITH and W. R. LONGLEY, PH.D.

First Course in Algebra
By H. E. HAWKES, PH.D., WILLIAM A. LUBY, A.B., and FRANK C. TOUTON, PH.B.

Second Course in Algebra
By H. E. HAWKES, PH.D., WILLIAM A. LUBY, A.B., and FRANK C. TOUTON, PH.B.

Elementary Analysis
By P. F. SMITH and W. A. GRANVILLE, PH.D

TEXT-BOOK ON THE STRENGTH OF MATERIALS

G B Upton
June 1912

BY

S. E. SLOCUM, B.E., PH.D.
PROFESSOR OF APPLIED MATHEMATICS IN THE UNIVERSITY OF CINCINNATI

AND

E. L. HANCOCK, M.S.
PROFESSOR OF APPLIED MECHANICS IN WORCESTER POLYTECHNIC INSTITUTE

REVISED EDITION

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

Five years of extensive use of this book, since the appearance of the first edition, have brought to the authors from various sources numerous suggestions relating to its improvement. In particular the authors wish to acknowledge their indebtedness to Professor Irving P. Church of Cornell University and to Professor George R. Chatburn of the University of Nebraska for their unfailing interest and frequent valuable suggestions.

To utilize the material so obtained, the text has been thoroughly revised. In making this revision the aim of the authors has been twofold: first, to keep the text abreast of the most recent practical developments of the subject; and second, to simplify the method of presentation so as to make the subject easily intelligible to the average technical student of junior grade, as well as to lessen the work of instruction.

Besides correcting the errors inevitable to a first edition, special attention has been given to amplifying the explanation wherever experience in using the book as a text has indicated it to be desirable. This applies especially to the articles on Poisson's ratio, the theorem of three moments, the calculation of the stress in curved members, the relation of Guest's and Rankine's formulas to the design of shafts subjected to combined stresses, etc.

Considerable new material has also been added. In Part I a set of tables has been placed at the beginning of the volume to facilitate numerical calculations. Other important additions are articles on the design of reënforced concrete beams, shrinkage and forced fits, the design of eccentrically loaded columns, the design and efficiency of riveted joints, the general theory of the torsion of springs, practical formulas for the collapse of tubes, and an extension of the method of least work to a wide variety of practical problems. This last includes

the derivation and application of the Fraenkel formula for the bending deflection of beams, and also a simple general formula for the shearing deflection of beams, never before published.

Nearly one hundred and fifty original problems have also been added to Part I. These problems are designed not merely to provide numerical exercises on the text, but have been selected throughout with the specific purpose of emphasizing the practical importance of the subject and extending the range of its application as widely as possible. Many of them are practical shop problems brought up by students in the coöperative engineering course at the University of Cincinnati.

In Part II the recent advances in the manufacture of steel have been given special attention, including the properties of vanadium steel, manganese steel, and high-speed steel. Reënforced concrete has also received a more adequate treatment, and the chapter on this subject has been thoroughly revised and modernized. The chapter on timber has also received an equally thorough revision, and considerable material on preservative processes has been added.

In both the first edition and the present revision, Part I, covering the analytical treatment of the subject, is the work of S. E. Slocum, and Part II, presenting the experimental or laboratory side, is the work of E. L. Hancock.

THE AUTHORS

CONTENTS

PART I — MECHANICS OF MATERIALS

CHAPTER I

ELASTIC PROPERTIES OF MATERIALS

CHAPTER II

FUNDAMENTAL RELATIONS BETWEEN STRESS AND DEFORMATION

CHAPTER III

ANALYSIS OF STRESS IN BEAMS

CHAPTER IV

FLEXURE OF BEAMS

CHAPTER V

COLUMNS AND STRUTS

CHAPTER VI

TORSION

NOTATION

The references are to articles.

A, B, C,	Constant coefficients, 49, 85.
C_1, C_2, etc.,	Constants of integration, 67, 85.
D,	Deflection, 67, 107.
E, E_s, E_c,	Young's modulus, 8.
F, F_1, F_2,	Area, 5.
G,	Modulus of shear, 33.
H,	Horse power, 99.
I, I_x, I_a, I_p, etc.,	Moment of inertia, 43.
J_{ik},	Influence numbers, 77.
K,	Coefficient of cubical expansion, 32.
L,	Coefficient of linear expansion, 19.
M, M_0, M_1, M_2,	External moment, 43.
N,	Statical moment, 47.
P, P', P_h, etc.,	Concentrated force, 5, 85, 86, 171.
Q,	Resultant shear, 53.
R, R_1, R_2,	Reactions of abutments, 50, 172. Resistance of soil, 168.
S,	Section modulus, 45, 170.
T,	Temperature change, 19.
V,	Volume, 32.
W,	Work, 73, 81.

---

a,	Semi-axis of ellipse, 49, 59, 103. Radius of shaft, 97.
b,	Semi-axis of ellipse, 49, 59, 103. Radius of shaft, 97. Breadth, 43, 104.
c,	Distance, 46, 47, 52, 170.
d,	Symbol of differentiation. Diameter of shaft, 99. Distance, 52, 67, 100, 168.
e,	Distance of extreme fiber from neutral axis, 43.
f,	Empirical constant, 89.
g,	Empirical constant, 89. Factor of safety, 172.
h,	Height, depth, 66.
k,	Coefficient of friction, 167. Constant, 115, 127, 132. Number, 229.

l,	Length, distance, 6, 47, 49, 85.
m,	Poisson's constant, 9.
n,	Abstract number, 92, 99, 170. Ratio, 66.
p, p_1, p', p_x, etc.,	Unit normal stress, 5, 23, 25.
p_e,	Equivalent normal stress, 35.
q, q_1, q', q_x, etc.,	Unit shear, 5, 23, 25.
r,	Radius, 46, 56, 96. Ratio, 161, 227.
s,	Unit deformation, 6.
t, t_x, t_a, etc.,	Radius of gyration, 46, 49.
u,	Curvilinear coördinate, 113. Bond, 229.
u_t,	Ultimate tensile strength, 117, 169.
u_c,	Ultimate compressive strength, 148, 149.
w,	Unit load, 51. Weight per cubic foot, 171.
x, y, z,	Variables.
$\bar{x}$, $\bar{y}$, $\bar{z}$,	Coördinates of center of gravity, 42.

α,	Angle, 25, 46, 171.
β,	Angle, 67, 75, 171.
δ, ϵ,	Empirical constants, 91.
ζ,	Angle, 171.
η,	Correction coefficient, 65.
θ,	Angle of twist, 96. Angle, 172.
κ,	Ratio between tensile and shearing strength, 57.
λ,	Arbitrary integer, 26, 85.
μ,	Constant, 99.
ν,	Empirical constant, 11, 92.
π,	Ratio of circumference to diameter.
ρ,	Radius of curvature, 67, 113.
σ,	Empirical constant, 11, 92.
Σ,	Symbol of summation, 25.
ϕ,	Angle of shear, 33, 96.
ω,	Angle of repose, 167.

TABLES OF PHYSICAL AND MATHEMATICAL CONSTANTS

TABLE I

1. Average Values of Physical Constants

Material	Ultimate Tensile Strength	Ultimate Compressive Strength	Ultimate Shearing Strength	Ultimate Flexural Strength (Modulus of Rupture)	Elastic Limit	Unit Elongation at Elastic Limit	Young's Modulus of Elasticity	Modulus of Shear (Modulus of Rigidity)	Weight	Coefficient of Linear Expansion
	lb./in.^{2}	lb./in.^{2}	lb./in.^{2}	lb./in.^{2}	lb./in.^{2}	inch	lb./in.^{2}	lb./in.^{2}	lb./ft.^{3}	1° F.
Hard steel	100,000	120,000	80,000	110,000	60,000	0.0012	30,000,000	12,000,000	490	.0000074
Structural steel	60,000	60,000	50,000	60,000	35,000	0.0012	30,000,000	12,000,000	490	.0000061
Wrought iron	50,000	50,000	40,000	50,000	25,000	0.0010	25,000,000	10,000,000	480	.0000068
Cast-iron tension	20,000	. . . .	20,000	35,000	6,000	0.0004	15,000,000	6,000,000	450	.0000063
" compression	. . . .	90,000	. .	. . .	20,000	. . .	. .	. . . .	. .	. . . .
Brass, drawn	43,000	11,000	. .	.		. .	14,500,000	5,000,000	530	. . . .
" cast	24,000	. .	. .	. . . . .		. .	9,500,000	.	520	. . . .
Copper, drawn	32,000	.	.	. . . . .		.	15,000,000	6,000,000	550	. . . .
cast	22,000	.	.	. .	.	.	12,000,000	. . . .	. . . .	. . . .
Timber, with grain	10,000	8,000	600	9,000	3,000	0.0020	1,500,000	. . . .	40	.0000028
" across grain	. . .	. .	3,000	. . .	. .	. .	. . .	400,000	. . . .	. . .
Concrete	300	3,000	1,000	700	1,000	. .	2,000,000	. . .	150	.0000055
Stone	. .	6,000	1,500	2,000	2,000	. . . .	6,000,000	1,800,000	160	.0000050
Brick	. . . .	3,000	1,000	800	1,000	. . . .	2,000,000	. . .	125	.0000050

2. Poisson's Ratio

Material	Average Values of $\frac{1}{m}$
Steel, hard	.295
" structural	.299
Iron	.277
Brass	.357
Copper	.340
Lead	.375
Zinc	.205

3. Factors of Safety

Material	Steady Stress: Buildings, etc.	Varying Stress: Bridges, etc.	Repeated or Reversed Stress: Machines
Steel, hard	5	8	15
" structural	4	6	10
Iron, wrought	4	6	10
" cast	6	10	20
Timber	8	10	15
Brick and stone	15	25	30

The only rational method of determining the factor of safety is to choose it sufficiently large to bring the working stress well within the elastic limit (see Article 14).

TABLE II

Properties of Various Sections

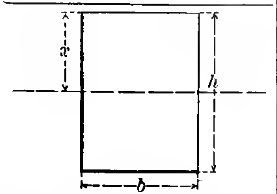
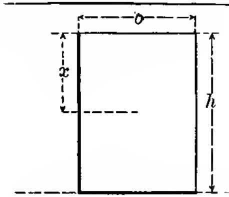
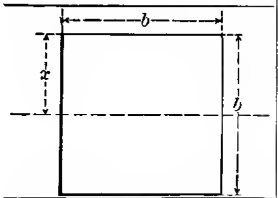
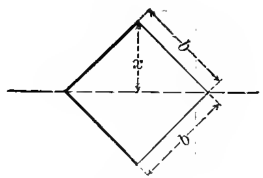
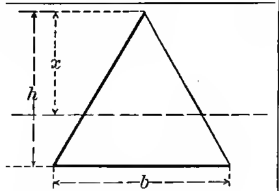
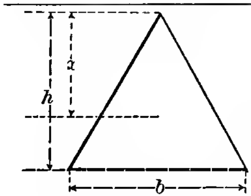
Shape	Area F	Location of Gravity Axis x	Static Moment of Inertia (Second Moment of Area) I	Section Modulus $S = \frac{I}{x}$
	bh	$x = \frac{h}{2}$	$\frac{bh^3}{12}$	$\frac{bh^2}{6}$
	bh	$x = \frac{h}{2}$	$\frac{bh^3}{3}$	
	b^2	$x = \frac{b}{2}$	$\frac{b^4}{12}$	$\frac{b^3}{6}$

TABLE II (*Continued*)

PROPERTIES OF VARIOUS SECTIONS

SHAPE	AREA F	LOCATION OF GRAVITY AXIS x	STATIC MOMENT OF INERTIA (SECOND MOMENT OF AREA) I	SECTION MODULUS $S=\frac{I}{x}$
	b^2	$x=\frac{b}{\sqrt{2}}=.707\,b$	$\frac{b^4}{12}$	$\frac{b^3}{6\sqrt{2}}=.118\,b^3$
	$\frac{1}{2}bh$	$x=\frac{2}{3}h$	$\frac{bh^3}{36}$	$\frac{bh^2}{24}$
	$\frac{1}{2}bh$	$x=\frac{2}{3}h$	$\frac{bh^3}{12}$	

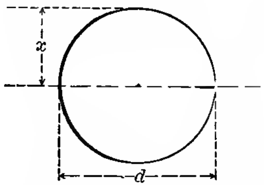
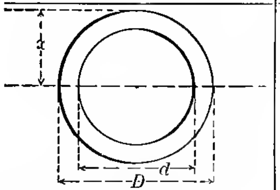
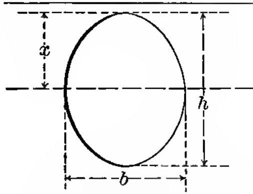
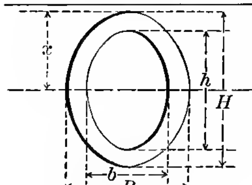
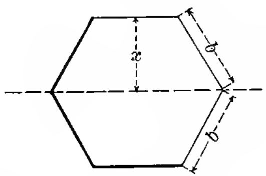
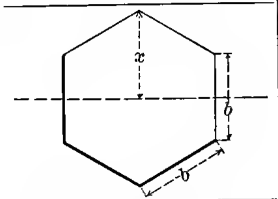
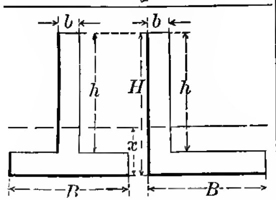
	$\frac{\pi d^2}{4} = .785\,d^2$	$x = \frac{d}{2}$	$\frac{\pi d^4}{64} = .049\,d^4$	$\frac{\pi d^3}{32} = .098\,d^3$
	$\frac{\pi\,(D^2 - d^2)}{4} = .785\,(D^2 - d^2)$	$x = \frac{D}{2}$	$\frac{\pi\,(D^4 - d^4)}{64} = .049\,(D^4 - d^4)$	$\frac{\pi\,(D^4 - d^4)}{32\,D}$ $= .098\left(D^3 - \frac{d^4}{D}\right)$
	$\frac{\pi bh}{4} = .785\,bh$	$x = \frac{h}{2}$	$\frac{\pi bh^3}{64} = .049\,bh^3$	$\frac{\pi bh^2}{32} = .098\,bh^2$
	$\frac{\pi}{4}(BH - bh) = .785(BH - bh)$	$x = \frac{H}{2}$	$\frac{\pi\,(BH^3 - bh^3)}{64}$ $= .049\,(BH^3 - bh^3)$	$\frac{\pi\,(BH^3 - bh^3)}{32\,H}$ $= .098\left(BH^2 - \frac{bh^3}{H}\right)$

TABLE II (*Continued*)

PROPERTIES OF VARIOUS SECTIONS

Shape	Area F	Location of Gravity Axis x	Static Moment of Inertia (Second Moment of Area) I	Section Modulus $S=\frac{I}{x}$
	$\frac{3\sqrt{3}}{2}b^2 = 2.6\,b^2$	$x = \frac{\sqrt{3}}{2}b = .866\,b$	$\frac{5\sqrt{3}}{16}b^4 = .5413\,b^4$	$\frac{5}{8}b^3$
	$\frac{3\sqrt{3}}{2}b^2 = 2.6\,b^2$	$x = b$	$\frac{5\sqrt{3}}{16}b^4 = .5413\,b^4$	$.5413\,b^3$
	$B(H-h)+bh$	$x = \frac{BH^2 - h(B-b)(2H-h)}{2[B(H-h)+bh]}$	$\frac{b(H-x)^3 + Bx^3 - (B-b)(x+h-H)^3}{3}$	$\frac{I}{H-x}$

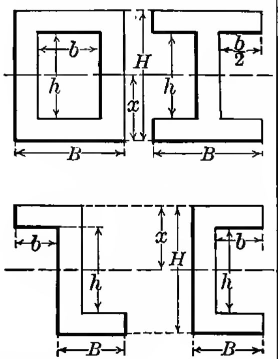
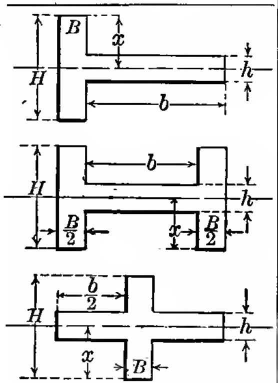
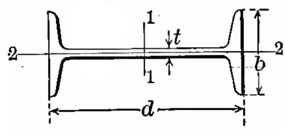
	$BH - bh$	$x = \frac{H}{2}$	$\frac{BH^3 - bh^3}{12}$	$\frac{BH^3 - bh^3}{6\,H}$
	$BH + bh$	$x = \frac{H}{2}$	$\frac{BH^3 + bh^3}{12}$	$\frac{BH^3 + bh^3}{6\,H}$

TABLE III

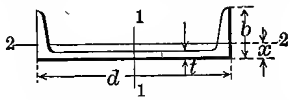
Properties of Standard I-Beams



Depth of Beam	Weight per Foot	Area of Section	Thick-ness of Web	Width of Flange	Moment of Inertia Axis 1-1	Section Modu-lus Axis 1-1	Radius of Gyra-tion Axis 1-1	Moment of Inertia Axis 2-2	Radius of Gyra-tion Axis 2-2
d		A	t	b	I	S	r	I′	r′
Inches	*Pounds*	*Sq. Inches*	*Inches*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*	*Inches*^{4}	*Inches*
3	5.5	1.63	.17	2.33	2.5	1.7	1.23	.46	.53
"	6.5	1.91	.26	2.42	2.7	1.8	1.19	.53	.52
"	7.5	2.21	.36	2.52	2.9	1.9	1.15	.60	.52
4	7.5	2.21	.19	2.66	6.0	3.0	1.64	.77	.59
"	8.5	2.50	.26	2.73	6.4	3.2	1.59	.85	.58
"	9.5	2.79	.34	2.81	6.7	3.4	1.54	.93	.58
"	10.5	3.09	.41	2.88	7.1	3.6	1.52	1.01	.57
5	9.75	2.87	.21	3.00	12.1	4.8	2.05	1.23	.65
"	12.25	3.60	.36	3.15	13.6	5.4	1.94	1.45	.63
"	14.75	4.34	.50	3.29	15.1	6.1	1.87	1.70	.63
6	12.25	3.61	.23	3.33	21.8	7.3	2.46	1.85	.72
"	14.75	4.34	.35	3.45	24.0	8.0	2.35	2.09	.69
"	17.25	5.07	.47	3.57	26.2	8.7	2.27	2.36	.68
7	15.0	4.42	.25	3.66	36.2	10.4	2.86	2.67	.78
"	17.5	5.15	.35	3.76	39.2	11.2	2.76	2.94	.76
"	20.0	5.88	.46	3.87	42.2	12.1	2.68	3.24	.74
8	17.75	5.33	.27	4.00	56.9	14.2	3.27	3.78	.84
"	20.25	5.96	.35	4.08	60.2	15.0	3.18	4.04	.82
"	22.75	6.69	.44	4.17	64.1	16.0	3.10	4.36	.81
"	25.25	7.43	.53	4.26	68.0	17.0	3.03	4.71	.80
9	21.0	6.31	.29	4.33	84.9	18.9	3.67	5.16	.90
"	25.0	7.35	.41	4.45	91.9	20.4	3.54	5.65	.88
"	30.0	8.82	.57	4.61	101.9	22.6	3.40	6.42	.85
"	35.0	10.29	.73	4.77	111.8	24.8	3.30	7.31	.84
10	25.0	7.37	.31	4.66	122.1	24.4	4.07	6.89	.97
"	30.0	8.82	.45	4.80	134.2	26.8	3.90	7.65	.93
"	35.0	10.29	.60	4.95	146.4	29.3	3.77	8.52	.91
"	40.0	11.76	.75	5.10	158.7	31.7	3.67	9.50	.90
12	32.5	9.26	.35	5.00	215.8	36.0	4.83	9.50	1.01
"	35.0	10.29	.44	5.09	228.3	38.0	4.71	10.07	.99
"	40.0	11.76	.56	5.21	245.9	41.0	4.57	10.95	.96
15	42.0	12.48	.41	5.50	441.8	58.9	5.95	14.62	1.08
"	45.0	13.24	.46	5.55	455.8	60.8	5.87	15.09	1.07
"	50.0	14.71	.56	5.65	483.4	64.5	5.73	16.04	1.04
"	55.0	16.18	.66	5.75	511.0	68.1	5.62	17.06	1.03
"	60.0	17.65	.75	5.84	538.6	71.8	5.52	18.17	1.01
18	55.0	15.93	.46	6.00	795.6	88.4	7.07	21.19	1.15
"	60.0	17.65	.56	6.10	841.8	93.5	6.91	22.38	1.13
"	65.0	19.12	.64	6.18	881.5	97.9	6.79	23.47	1.11
"	70.0	20.59	.72	6.26	921.2	102.4	6.69	24.62	1.09
20	65.0	19.08	.50	6.25	1169.5	117.0	7.83	27.86	1.21
"	70.0	20.59	.58	6.33	1219.8	122.0	7.70	29.04	1.19
"	75.0	22.06	.65	6.40	1268.8	126.9	7.58	30.25	1.17
24	80.0	23.32	.50	7.00	2087.2	173.9	9.46	42.86	1.36
"	85.0	25.00	.57	7.07	2167.8	180.7	9.31	44.35	1.33
"	90.0	26.47	.63	7.13	2238.4	186.5	9.20	45.70	1.31
"	95.0	27.94	.69	7.19	2309.0	192.4	9.09	77.10	1.30
"	100.0	29.41	.75	7.25	2379.6	198.3	8.99	48.55	1.28

TABLE IV

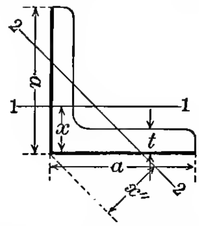
PROPERTIES OF STANDARD CHANNELS



DEPTH OF CHANNEL	WEIGHT PER FOOT	AREA OF SECTION	THICKNESS OF WEB	WIDTH OF FLANGE	MOMENT OF INERTIA AXIS 1-1	SECTION MODULUS AXIS 1-1	RADIUS OF GYRATION AXIS 1-1	MOMENT OF INERTIA AXIS 2-2	SECTION MODULUS AXIS 2-2	RADIUS OF GYRATION AXIS 2-2	DISTANCE OF CENTER OF GRAVITY FROM OUTSIDE OF WEB
d		A	t	b	I	S	r	I′	S′	r′	x
Inches	*Pounds*	*Sq. In.*	*Inches*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*	*Inches*	*Inches*^{3}	*Inches*	*Inches*
3	4.00	1.19	.17	1.41	1.6	1.1	1.17	.20	.21	.41	.44
“	5.00	1.47	.26	1.50	1.8	1.2	1.12	.25	.24	.41	.44
“	6.00	1.76	.36	1.60	2.1	1.4	1.08	.31	.27	.42	.46
4	5.25	1.55	.18	1.58	3.8	1.9	1.56	.32	.29	.45	.46
“	6.25	1.84	.25	1.65	4.2	2.1	1.51	.38	.32	.45	.46
“	7.25	2.13	.33	1.73	4.6	2.3	1.46	.44	.35	.46	.46
5	6.50	1.95	.19	1.75	7.4	3.0	1.95	.48	.38	.50	.49
“	9.00	2.65	.33	1.89	8.9	3.5	1.83	.64	.45	.49	.48
“	11.50	3.38	.48	2.04	10.4	4.2	1.75	.82	.54	.49	.51
6	8.00	2.38	.20	1.92	13.0	4.3	2.34	.70	.50	.54	.52
“	10.50	3.09	.32	2.04	15.1	5.0	2.21	.88	.57	.53	.50
“	13.00	3.82	.44	2.16	17.3	5.8	2.13	1.07	.65	.53	.52
“	15.50	4.56	.56	2.28	19.5	6.5	2.07	1.28	.74	.53	.55
7	9.75	2.85	.21	2.09	21.1	6.0	2.72	.98	.63	.59	.55
“	12.25	3.60	.32	2.20	24.2	6.9	2.59	1.19	.71	.57	.53
“	14.75	4.34	.42	2.30	27.2	7.8	2.50	1.40	.79	.57	.53
“	17.25	5.07	.53	2.41	30.2	8.6	2.44	1.62	.87	.56	.55
“	19.75	5.81	.63	2.51	33.2	9.5	2.39	1.85	.96	.56	.58
8	11.25	3.35	.22	2.26	32.3	8.1	3.10	1.33	.79	.63	.58
“	13.75	4.04	.31	2.35	36.0	9.0	2.98	1.55	.87	.62	.56
“	16.25	4.78	.40	2.44	39.9	10.0	2.89	1.78	.95	.61	.56
“	18.75	5.51	.49	2.53	43.8	11.0	2.82	2.01	1.02	.60	.57
“	21.25	6.25	.58	2.62	47.8	11.9	2.76	2.25	1.11	.60	.59
9	13.25	3.89	.23	2.43	47.3	10.5	3.49	1.77	.97	.67	.61
“	15.00	4.41	.29	2.49	50.9	11.3	3.40	1.95	1.03	.66	.59
“	20.00	5.88	.45	2.65	60.8	13.5	3.21	2.45	1.19	.65	.58
“	25.00	7.35	.61	2.81	70.7	15.7	3.10	2.98	1.36	.64	.62
10	15.00	4.46	.24	2.60	66.9	13.4	3.87	2.30	1.17	.72	.64
“	20.00	5.88	.38	2.74	78.7	15.7	3.66	2.85	1.34	.70	.61
“	25.00	7.35	.53	2.89	91.0	18.2	3.52	3.40	1.50	.68	.62
“	30.00	8.82	.68	3.04	103.2	20.6	3.42	3.99	1.67	.67	.65
“	35.00	10.29	.82	3.18	115.5	23.1	3.35	4.66	1.87	.67	.69
12	20.50	6.03	.28	2.94	128.1	21.4	4.61	3.91	1.75	.81	.70
“	25.00	7.35	.39	3.05	144.0	24.0	4.43	4.53	1.91	.78	.68
“	30.00	8.82	.51	3.17	161.6	26.9	4.28	5.21	2.09	.77	.68
“	35.00	10.29	.64	3.30	179.3	29.9	4.17	5.90	2.27	.76	.69
“	40.00	11.76	.76	3.42	196.9	32.8	4.09	6.63	2.46	.75	.72
15	33.00	9.90	.40	3.40	312.6	41.7	5.62	8.23	3.16	.91	.79
“	35.00	10.29	.43	3.43	319.9	42.7	5.57	8.48	3.22	.91	.79
“	40.00	11.76	.52	3.52	347.5	46.3	5.44	9.39	3.43	.89	.78
“	45.00	13.24	.62	3.62	375.1	50.0	5.32	10.29	3.63	.88	.79
“	50.00	14.71	.72	3.72	402.7	53.7	5.23	11.22	3.85	.87	.80
“	55.00	16.18	.82	3.82	430.2	57.4	5.16	12.19	4.07	.87	.82

TABLE V

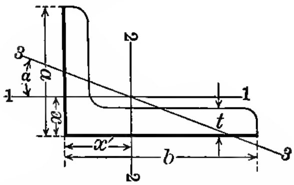
Properties of Standard Angles, Equal Legs



Dimensions	Thickness	Weight per Foot	Area of Section	Distance of Center of Gravity from Back of Flange	Moment of Inertia Axis 1-1	Section Modulus Axis 1-1	Radius of Gyration Axis 1-1	Distance of Center of Gravity from External Apex on Line inclined at 45° to Flange	Least Moment of Inertia Axis 2-2	Section Modulus Axis 2-2	Least Radius of Gyration Axis 2-2
Inches	*Inches*	*Pounds*	*Sq. In.*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*
$\frac{3}{4} \times \frac{3}{4}$	$\frac{1}{8}$	.58	.17	.23	.009	.017	.22	.33	.004	.011	.14
1×1	$\frac{1}{8}$	.80	.23	.30	.022	.031	.30	.42	.009	.021	.19
"	$\frac{1}{4}$	1.49	.44	.34	.037	.056	.29	.48	.016	.034	.19
$1\frac{1}{4} \times 1\frac{1}{4}$	$\frac{1}{8}$	1.02	.30	.36	.044	.049	.38	.51	.018	.035	.24
"	$\frac{1}{4}$	1.91	.56	.40	.077	.091	.37	.57	.033	.057	.24
$1\frac{1}{2} \times 1\frac{1}{2}$	$\frac{1}{4}$	2.34	.69	.47	.14	.134	.45	.66	.058	.088	.29
"	$\frac{3}{8}$	3.35	.98	.51	.19	.188	.44	.72	.082	.114	.29
$1\frac{3}{4} \times 1\frac{3}{4}$	$\frac{1}{4}$	2.77	.81	.53	.23	.19	.53	.75	.094	.13	.34
"	$\frac{3}{8}$	3.98	1.17	.57	.31	.26	.51	.81	.133	.16	.34
2×2	$\frac{1}{4}$	3.19	.94	.59	.35	.25	.61	.84	.14	.17	.39
"	$\frac{3}{8}$	4.62	1.36	.64	.48	.35	.59	.90	.20	.22	.39
$2\frac{1}{2} \times 2\frac{1}{2}$	$\frac{1}{4}$	4.0	1.19	.72	.70	.39	.77	1.01	.29	.28	.49
"	$\frac{3}{8}$	5.9	1.73	.76	.98	.57	.75	1.08	.41	.38	.48
"	$\frac{1}{2}$	7.7	2.25	.81	1.23	.72	.74	1.14	.52	.46	.48
3×3	$\frac{1}{4}$	4.9	1.44	.84	1.24	.58	.93	1.19	.50	.42	.59
"	$\frac{3}{8}$	7.2	2.11	.89	1.76	.83	.91	1.26	.72	.57	.58
"	$\frac{1}{2}$	9.4	2.75	.93	2.22	1.07	.90	1.32	.92	.70	.58
"	$\frac{5}{8}$	11.4	3.36	.98	2.62	1.30	.88	1.38	1.12	.81	.58
$3\frac{1}{2} \times 3\frac{1}{2}$	$\frac{3}{8}$	8.4	2.48	1.01	2.87	1.15	1.07	1.43	1.16	.81	.68
"	$\frac{1}{2}$	11.1	3.25	1.06	3.64	1.49	1.06	1.50	1.50	1.00	.68
"	$\frac{5}{8}$	13.5	3.98	1.10	4.33	1.81	1.04	1.56	1.82	1.17	.68
"	$\frac{3}{4}$	15.9	4.69	1.15	4.96	2.11	1.03	1.62	2.13	1.31	.67
4×4	$\frac{3}{8}$	9.7	2.86	1.14	4.36	1.52	1.23	1.61	1.77	1.10	.79
"	$\frac{1}{2}$	12.8	3.75	1.18	5.56	1.97	1.22	1.67	2.28	1.36	.78
"	$\frac{5}{8}$	15.7	4.61	1.23	6.66	2.40	1.20	1.74	2.76	1.59	.77
"	$\frac{3}{4}$	18.5	5.44	1.27	7.66	2.81	1.19	1.80	3.23	1.80	.77
6×6	$\frac{1}{2}$	19.6	5.75	1.68	19.91	4.61	1.86	2.38	8.04	3.37	1.18
"	$\frac{5}{8}$	24.2	7.11	1.73	24.16	5.66	1.84	2.45	9.81	4.01	1.17
"	$\frac{3}{4}$	28.7	8.44	1.78	28.15	6.66	1.83	2.51	11.52	4.59	1.17
"	$\frac{7}{8}$	33.1	9.73	1.82	31.92	7.63	1.81	2.57	13.17	5.12	1.16

TABLE V

Properties of Standard Angles, Unequal Legs



Dimensions	Thickness	Weight per Foot	Area of Section	Distance of Center of Gravity from Back of Longer Flange	Moment of Inertia Axis 1-1	Section Modulus Axis 1-1	Radius of Gyration Axis 1-1	Distance of Center of Gravity from Back of Shorter Flange	Moment of Inertia Axis 2-2	Section Modulus Axis 2-2	Radius of Gyration Axis 2-2
Inches	*Inches*	*Pounds*	*Sq. In.*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*	*Inches*	*Inches*^{4}	*Inches*^{3}	*Inches*
$2\frac{1}{2} \times 2$	$\frac{1}{4}$	3.6	1.06	.54	.37	.25	.59	.79	.65	.38	.78
"	$\frac{3}{8}$	5.3	1.55	.58	.51	.36	.58	.83	.91	.55	.77
"	$\frac{1}{2}$	6.8	2.00	.63	.64	.46	.56	.88	1.14	.70	.75
$3 \times 2\frac{1}{2}$	$\frac{1}{4}$	4.5	1.31	.66	.74	.40	.75	.91	1.17	.56	.95
"	$\frac{3}{8}$	6.5	1.92	.71	1.04	.58	.74	.96	1.66	.81	.93
"	$\frac{1}{2}$	8.5	2.50	.75	1.30	.74	.72	1.00	2.08	1.04	.91
$3\frac{1}{2} \times 2\frac{1}{2}$	$\frac{1}{4}$	4.9	1.44	.61	.78	.41	.74	1.11	1.80	.75	1.12
"	$\frac{3}{8}$	7.2	2.11	.66	1.09	.59	.72	1.16	2.56	1.09	1.10
"	$\frac{1}{2}$	9.4	2.75	.70	1.36	.76	.70	1.20	3.24	1.41	1.09
"	$\frac{5}{8}$	11.4	3.36	.75	1.61	.92	.69	1.25	3.85	1.71	1.07
$3\frac{1}{2} \times 3$	$\frac{3}{8}$	7.8	2.30	.83	1.85	.85	.90	1.08	2.72	1.13	1.09
"	$\frac{1}{2}$	10.2	3.00	.88	2.33	1.10	.88	1.13	3.45	1.45	1.07
"	$\frac{5}{8}$	12.5	3.67	.92	2.76	1.33	.87	1.17	4.11	1.76	1.06
"	$\frac{3}{4}$	14.7	4.31	.96	3.15	1.54	.85	1.21	4.70	2.05	1.04
4×3	$\frac{3}{8}$	8.5	2.48	.78	1.92	.87	.88	1.28	3.96	1.46	1.26
"	$\frac{1}{2}$	11.1	3.25	.83	2.42	1.12	.86	1.33	5.05	1.89	1.25
"	$\frac{5}{8}$	13.6	3.98	.87	2.87	1.35	.85	1.37	6.03	2.30	1.23
"	$\frac{3}{4}$	15.9	4.69	.92	3.28	1.57	.84	1.42	6.93	2.68	1.22
5×3	$\frac{3}{8}$	9.7	2.86	.70	2.04	.89	.84	1.70	7.37	2.24	1.61
"	$\frac{1}{2}$	12.8	3.75	.75	2.58	1.15	.83	1.75	9.45	2.91	1.59
"	$\frac{5}{8}$	15.7	4.61	.80	3.06	1.39	.82	1.80	11.37	3.55	1.57
"	$\frac{3}{4}$	18.5	5.44	.84	3.51	1.62	.80	1.84	13.15	4.16	1.55
$5 \times 3\frac{1}{2}$	$\frac{3}{8}$	10.4	3.05	.86	3.18	1.21	1.02	1.61	7.78	2.29	1.60
"	$\frac{1}{2}$	13.6	4.00	.91	4.05	1.56	1.01	1.66	9.99	2.99	1.58
"	$\frac{5}{8}$	16.7	4.92	.95	4.83	1.90	.99	1.70	12.03	3.65	1.56
"	$\frac{3}{4}$	19.8	5.81	1.00	5.55	2.22	.98	1.75	13.92	4.28	1.55
"	$\frac{7}{8}$	22.7	6.67	1.04	6.21	2.52	.96	1.79	15.67	4.88	1.53
$6 \times 3\frac{1}{2}$	$\frac{3}{8}$	11.6	3.42	.79	3.34	1.23	.99	2.04	12.86	3.24	1.94
"	$\frac{1}{2}$	15.3	4.50	.83	4.25	1.59	.97	2.08	16.59	4.24	1.92
"	$\frac{5}{8}$	18.9	5.55	.88	5.08	1.94	.96	2.13	20.08	5.19	1.90
"	$\frac{3}{4}$	22.3	6.56	.93	5.84	2.27	.94	2.18	23.34	6.10	1.89
"	$\frac{7}{8}$	25.7	7.55	.97	6.55	2.59	.93	2.22	26.39	6.98	1.87
6×4	$\frac{3}{8}$	12.3	3.61	.94	4.90	1.60	1.17	1.94	13.47	3.32	1.93
"	$\frac{1}{2}$	16.2	4.75	.99	6.27	2.08	1.15	1.99	17.40	4.33	1.91
"	$\frac{5}{8}$	19.9	5.86	1.03	7.52	2.54	1.13	2.03	21.07	5.31	1.90
"	$\frac{3}{4}$	23.6	6.94	1.08	8.68	2.97	1.12	2.08	24.51	6.25	1.88
"	$\frac{7}{8}$	27.2	7.98	1.12	9.75	3.39	1.11	2.12	27.73	7.15	1.86

TABLE VI

MOMENTS OF INERTIA AND SECTION MODULI: RECTANGULAR CROSS SECTION

Breadth b	Height h	Moment of Inertia $I=\frac{bh^3}{12}$	Section Modulus $S=\frac{bh^2}{6}$
1	1	.0833	.166
	2	.66	.66
	3	2.25	1.5
	4	5.33	2.66
	5	10.42	4.16
	6	18	6
	7	28.58	8.16
	8	42.66	10.66
	9	60.75	13.5
	10	83.33	16.66
	11	110.92	20.16
	12	144	24
2	2	1.33	1.33
	3	4.5	3
	4	10.66	5.33
	5	20.83	8.33
	6	36	12
	7	57.16	16.33
	8	85.33	21.33
	9	121.5	27
	10	166.66	33.33
	11	221.85	40.33
	12	288	48
3	3	6.75	4.5
	4	16	8
	5	31.25	12.5
	6	54	18
	7	85.75	24.5
	8	128	32
	9	182.25	40.5
	10	250	50
	11	332.75	60.5
	12	432	72

Breadth b	Height h	Moment of Inertia $I=\frac{bh^3}{12}$	Section Modulus $S=\frac{bh^2}{6}$
4	4	21.33	10.66
	5	41.66	16.66
	6	72	24
	7	114.33	32.66
	8	170.66	42.66
	9	243	54
	10	333.33	66.66
	11	443.66	80.66
	12	576	96
5	5	52.08	20.83
	6	90	30
	7	142.92	40.83
	8	213.33	53.33
	9	303.75	67.5
	10	416.66	83.33
	11	554.58	100.83
	12	720	120
6	6	108	36
	7	171.5	49
	8	256	64
	9	364.5	81
	10	500	100
	11	665.5	121
	12	864	144
7	7	200.08	57.16
	8	298.66	74.66
	9	425.25	94.5
	10	583.33	116.66
	11	776.42	141.16
	12	1008	168
	13	1281.58	197.16
	14	1600.66	228.66

Breadth b	Height h	Moment of Inertia $I=\frac{bh^3}{12}$	Section Modulus $S=\frac{bh^2}{6}$
8	8	341.33	85.33
	9	486	108
	10	666.66	133.33
	11	887.33	161.33
	12	1152	192
	13	1464.66	225.33
	14	1829.33	261.33
	15	2250	300
	16	2730.66	341.33
9	9	546.75	121.5
	10	750	150
	11	998.25	181.5
	12	1296	216
	13	1647.75	253.5
	14	2058	294
	15	2531.25	337.5
	16	3072	384
	17	3684.75	433.5
	18	4374	486
10	10	833.33	166.66
	11	1109.16	201.66
	12	1440	240
	13	1830.83	281.66
	14	2286.66	326.66
	15	2810	375
	16	3413.33	426.66
	17	4094.17	481.66
	18	4860	540
	19	5715.83	601.66
	20	6666.66	666.66

TABLE VII

Moments of Inertia and Section Moduli: Circular Cross Section

Diameter	Moment of Inertia	Section Modulus	Diameter	Moment of Inertia	Section Modulus	Diameter	Moment of Inertia	Section Modulus
1	.0491	.0982	35	73,662	4,209	69	1,112,660	32,251
2	.7854	.7854	36	82,448	4,580	70	1,178,588	33,674
3	3.976	2.651	37	91,998	4,973	71	1,247,393	35,138
4	12.57	6.283	38	102,354	5,387	72	1,319,167	36,644
5	30.68	12.27	39	113,561	5,824	73	1,393,995	38,192
6	63.62	21.21	40	125,664	6,283	74	1,471,963	39,783
7	117.9	33.67	41	138,709	6,766	75	1,553,156	41,417
8	201.1	50.27	42	152,745	7,274	76	1,637,662	43,096
9	322.1	71.57	43	167,820	7,806	77	1,725,571	44,820
10	490.9	98.17	44	183,984	8,363	78	1,816,972	46,589
11	718.7	130.7	45	201,289	8,946	79	1,911,967	48,404
12	1,018	169.6	46	219,787	9,556	80	2,010,619	50,265
13	1,402	215.7	47	239,531	10,193	81	2,113,051	52,174
14	1,886	269.4	48	260,576	10,857	82	2,219,347	54,130
15	2,485	331.3	49	282,979	11,550	83	2,329,605	56,135
16	3,217	402.1	50	306,796	12,270	84	2,443,920	58,189
17	4,100	482.3	51	332,086	13,023	85	2,562,392	60,292
18	5,153	572.6	52	358,908	13,804	86	2,685,120	62,445
19	6,397	673.4	53	387,323	14,616	87	2,812,205	64,648
20	7,854	785.4	54	417,393	15,459	88	2,943,748	66,903
21	9,547	909.2	55	449,180	16,334	89	3,079,853	69,210
22	11,499	1,045	56	482,750	17,241	90	3,220,623	71,569
23	13,737	1,194	57	518,166	18,181	91	3,366,165	73,982
24	16,286	1,357	58	555,497	19,155	92	3,516,586	76,448
25	19,175	1,534	59	594,810	20,163	93	3,671,992	78,968
26	22,432	1,726	60	636,172	21,206	94	3,832,492	81,542
27	26,087	1,932	61	679,651	22,284	95	3,998,198	84,173
28	30,172	2,155	62	725,332	23,398	96	4,169,220	86,859
29	34,719	2,394	63	773,272	24,548	97	4,345,671	89,601
30	39,761	2,651	64	823,550	25,736	98	4,527,664	92,401
31	45,333	2,925	65	876,240	26,961	99	4,715,315	95,259
32	51,472	3,217	66	931,420	28,225	100	4,908,727	98,175
33	58,214	3,528	67	989,166	29,527			
34	65,597	3,859	68	1,049,556	30,869			

TABLE VIII

Four-Place Logarithms of Numbers

1	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	0	1	2	3	4	5	6	7	8	9

50	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0	1	2	3	4	5	6	7	8	9

TABLE IX

Conversion of Logarithms

Reduction of Common Logarithms to Natural Logarithms

Rule for using Table. Divide the given common logarithm into periods of two digits and take from the table the corresponding numbers, having regard to their value as decimals. The sum will be the required natural logarithm.

Example. Find the natural logarithm corresponding to the common logarithm .497149.

Common Logarithms	Natural Logarithms
.49	1.1282667
.0071	.016348354
.000049	.00011282667
.497149	1.14472788067

Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm	Common Logarithm	Natural Logarithm
1	2.30259	26	59.86721	51	117.43184	76	174.99647
2	4.60517	27	62.16980	52	119.73442	77	177.29905
3	6.90776	28	64.47238	53	122.03701	78	179.60164
4	9.21034	29	66.77497	54	124.33959	79	181.90422
5	11.51293	30	69.07755	55	126.64218	80	184.20681
6	13.81551	31	71.38014	56	128.94477	81	186.50939
7	16.11810	32	73.68272	57	131.24735	82	188.81198
8	18.42068	33	75.98531	58	133.54994	83	191.11456
9	20.73327	34	78.28789	59	135.85252	84	193.41715
10	23.02585	35	80.59048	60	138.15511	85	195.71973
11	25.32844	36	82.89306	61	140.45769	86	198.02232
12	27.63102	37	85.19565	62	142.76028	87	200.32490
13	29.93361	38	87.49823	63	145.06286	88	202.62749
14	32.23619	39	89.80082	64	147.36545	89	204.93007
15	34.53878	40	92.10340	65	149.66803	90	207.23266
16	36.84136	41	94.40599	66	151.97062	91	209.53524
17	39.14395	42	96.70857	67	154.27320	92	211.83783
18	41.44653	43	99.01116	68	156.57579	93	214.14041
19	43.74912	44	101.31374	69	158.87837	94	216.44300
20	46.05170	45	103.61633	70	161.18096	95	218.74558
21	48.35429	46	105.91891	71	163.48354	96	221.04817
22	50.65687	47	108.22150	72	165.78613	97	223.35075
23	52.95946	48	110.52408	73	168.08871	98	225.65334
24	55.26204	49	112.82667	74	170.39130	99	227.95592
25	57.56463	50	115.12925	75	172.69388	100	230.25851

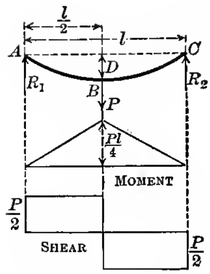
TABLE X

Functions of Angles

Angle	Sin	Tan	Sec	Cosec	Cot	Cos	
0	0.	0.	.0	∞	∞	1.	90
1	0.0175	0.0175	1.0001	57.299	57.290	0.9998	89
2	.0349	.0349	1.0006	28.654	28.636	.9994	88
3	.0523	.0524	1.0014	19.107	19.081	.9986	87
4	.0698	.0699	1.0024	14.336	14.301	.9976	86
5	.0872	.0875	1.0038	11.474	11.430	.9962	85
6	0.1045	0.1051	1.0055	9.5668	9.5144	0.9945	84
7	.1219	.1228	1.0075	8.2055	8.1443	.9925	83
8	.1392	.1405	1.0098	7.1853	7.1154	.9903	82
9	.1564	.1584	1.0125	6.3925	6.3138	.9877	81
10	.1736	.1763	1.0154	5.7588	5.6713	.9848	80
11	0.1908	0.1944	1.0187	5.2408	5.1446	0.9816	79
12	.2079	.2126	1.0223	4.8097	4.7046	.9781	78
13	.2250	.2309	1.0263	4.4454	4.3315	.9744	77
14	.2419	.2493	1.0306	4.1336	4.0108	.9703	76
15	.2588	.2679	1.0353	3.8637	3.7321	.9659	75
16	0.2756	0.2867	1.0403	3.6280	3.4874	0.9613	74
17	.2924	.3057	1.0457	3.4203	3.2709	.9563	73
18	.3090	.3249	1.0515	3.2361	3.0777	.9511	72
19	.3256	.3443	1.0576	3.0716	2.9042	.9455	71
20	.3420	.3640	1.0642	2.9238	2.7475	.9397	70
21	0.3584	0.3839	1.0712	2.7904	2.6051	0.9336	69
22	.3746	.4040	1.0785	2.6695	2.4751	.9272	68
23	.3907	.4245	1.0864	2.5593	2.3559	.9205	67
24	.4067	.4452	1.0946	2.4586	2.2460	.9135	66
25	.4226	.4663	1.1034	2.3662	2.1445	.9063	65
26	0.4384	0.4877	1.1126	2.2812	2.0503	0.8988	64
27	.4540	.5095	1.1223	2.2027	1.9626	.8910	63
28	.4695	.5317	1.1326	2.1301	1.8807	.8829	62
29	.4848	.5543	1.1434	2.0627	1.8040	.8746	61
30	.5000	.5774	1.1547	2.0000	1.7321	.8660	60
31	0.5150	0.6009	1.1666	1.9416	1.6643	0.8572	59
32	.5299	.6249	1.1792	1.8871	1.6003	.8480	58
33	.5446	.6494	1.1924	1.8361	1.5399	.8387	57
34	.5592	.6745	1.2062	1.7883	1.4826	.8290	56
35	.5736	.7002	1.2208	1.7435	1.4281	.8192	55
36	0.5878	0.7265	1.2361	1.7013	1.3764	0.8090	54
37	.6018	.7536	1.2521	1.6616	1.3270	.7986	53
38	.6157	.7813	1.2690	1.6243	1.2799	.7880	52
39	.6293	.8098	1.2868	1.5890	1.2349	.7771	51
40	.6428	.8391	1.3054	1.5557	1.1918	.7660	50
41	0.6561	0.8693	1.3250	1.5243	1.1504	0.7547	49
42	.6691	.9004	1.3456	1.4945	1.1106	.7431	48
43	.6820	.9325	1.3673	1.4663	1.0724	.7314	47
44	.6947	.9657	1.3902	1.4396	1.0355	.7193	46
45	.7071	1.	1.4142	1.4142	1.	.7071	45
	Cos	Cot	Cosec	Sec	Tan	Sin	Angle

TABLE XI

Bending Moment and Shear Diagrams

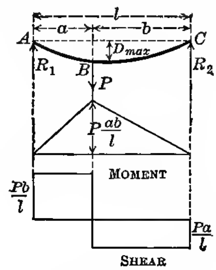


$$R_1 = R_2 = \frac{P}{2}.$$

$$M_a = M_c = 0.$$

$$M_b = \frac{Pl}{4}.$$

$$D = \frac{Pl^3}{48\,EI}.$$

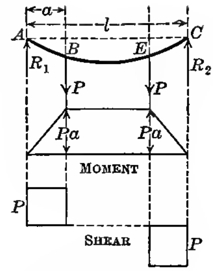


$$D_{max} = \frac{Pab\,(2l-a)}{9EI\,l}\sqrt{\frac{a\,(2l-a)}{3}}$$

$$R_1 = \frac{Pb}{l}. \qquad R_2 = \frac{Pa}{l}.$$

$$M_a = M_c = 0.$$

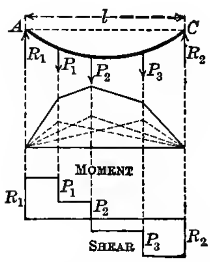
$$M_b = P\,\frac{ab}{l}.$$



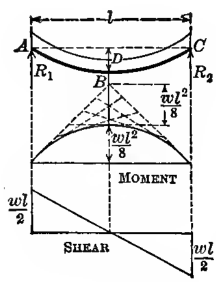
$$R_1 = R_2 = P.$$

$$M_a = M_c = 0.$$

$$M_b = M_E = P_a.$$



$$M_a = M_c = 0.$$

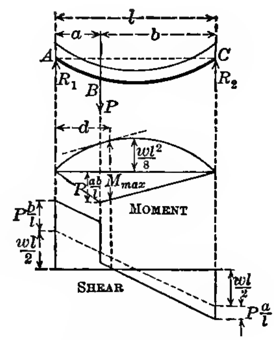


$$R_1 = R_2 = \frac{wl}{2}.$$

$$M_a = M_c = 0.$$

$$M_b = \frac{wl^2}{8}.$$

$$D = \frac{5\,wl^4}{384\,EI}.$$

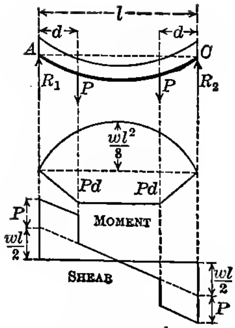


$$R_1 = \frac{wl}{2} + \frac{Pb}{l}.$$

$$R_2 = \frac{wl}{2} + \frac{Pa}{l}.$$

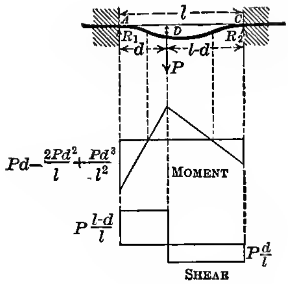
$$d = \frac{l}{2} - \frac{Pd}{wl}.$$

$$M_{\max} = R_1 d - P\,(d - a).$$



$$R_1 = R_2 = \frac{wl}{2} + P.$$

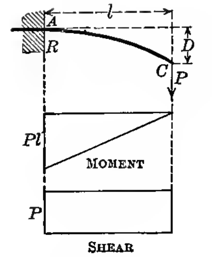
$$M_{\max} = \frac{wl^2}{8} + Pd.$$



$$R_1 = P + \frac{2\,Pd^3}{l^3} - \frac{3\,Pd^2}{l^2}.$$

$$M_A = Pd - \frac{2\,Pd^2}{l} + \frac{Pd^3}{l^2}.$$

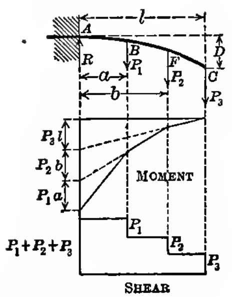
$$D = \frac{Pd^3\,(l-d)^3}{3\,EIl^3}.$$



$$R = P.$$

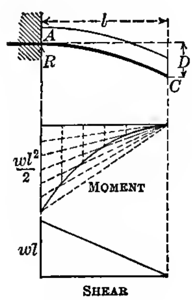
$$M_A = -\,Pl.$$

$$D = \frac{Pl^3}{3\,EI}.$$



$$R = P_1 + P_2 + P_3.$$

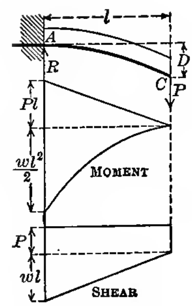
$$M_A = -\,P_1a - P_2b - P_3l.$$



$$R = wl.$$

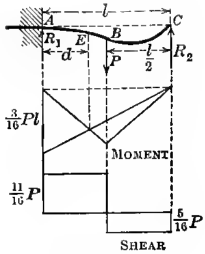
$$M_A = \frac{-\,wl^2}{2}.$$

$$D = \frac{wl^2}{8\,EI}.$$

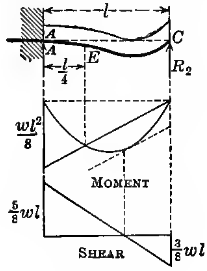


$$R = wl + P.$$

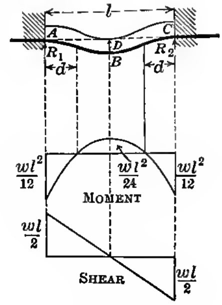
$$M_A = -\,Pl - \frac{wl^2}{2}.$$



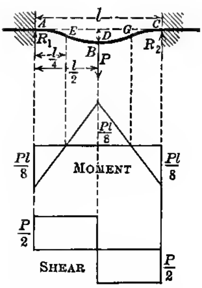
$$R_1 = \tfrac{11}{16}P.$$
$$R_2 = \tfrac{5}{16}P.$$
$$M_A = -\tfrac{3}{16}Pl.$$
$$M_B = \tfrac{5}{32}Pl.$$
$$d = \tfrac{3}{11}l.$$



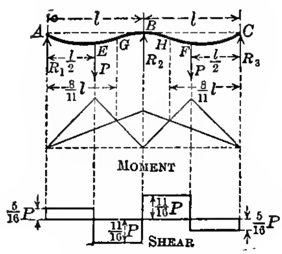
$$R_1 = \tfrac{5}{8}wl.$$
$$R_2 = \tfrac{3}{8}wl.$$
$$M_A = -\frac{wl^2}{8}.$$



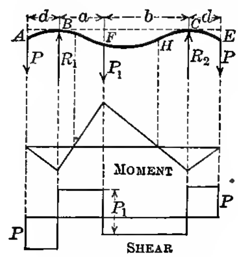
$$R_1 = R_2 = \frac{wl}{2}.$$
$$M_A = M_c = \frac{wl^2}{12}.$$
$$M_B = \frac{wl^2}{24}.$$
$$D = \frac{wl^4}{384\,EI}.$$



$$R_1 = R_2 = \frac{P}{2}.$$
$$M_A = M_c = -\frac{Pl}{8}.$$
$$M_B = \frac{Pl}{8}.$$
$$D = \frac{Pl^3}{182\,EI}.$$

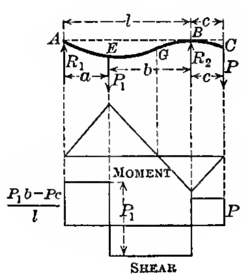


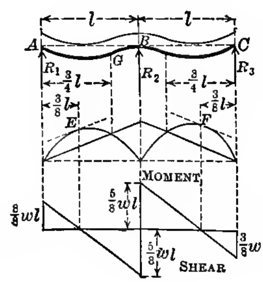
$$R_1 = R_3 = \tfrac{5}{16}P.$$
$$R_2 = \tfrac{11}{8}P.$$
$$M_a = 0.$$
$$M_c = 0.$$
$$M_E = \tfrac{5}{32}Pl.$$
$$M_B = -\tfrac{3}{16}Pl.$$
$$M_F = \tfrac{5}{32}Pl.$$



$$R_1 = P + P_1\frac{b}{l}.$$
$$R_2 = P + P_1\frac{a}{l}.$$
$$M_b = -Pd.$$
$$M_c = -Pd.$$
$$M_F = P_1\frac{ab}{l} - Pd.$$

$$R_1 = \frac{P_1 b - Pc}{l}.$$
$$R_2 = \frac{P_1 a + P(l+c)}{l}.$$
$$M_a = 0.$$
$$M_b = -Pc.$$
$$M_E = \frac{(P_1 b - Pc)\,a}{l}.$$





STRENGTH OF MATERIALS

PART I

MECHANICS OF MATERIALS

PART I

MECHANICS OF MATERIALS

CHAPTER I

ELASTIC PROPERTIES OF MATERIALS

1. Introductory. In mechanics all bodies considered are assumed to be perfectly rigid; that is to say, it is assumed that no matter what system of forces acts on a body, the distance between any two points of the body remains unchanged.

It has been found by experiment, however, that the behavior of natural bodies does not verify this assumption. Thus experiment shows that when a body formed of any substance whatever is acted upon by external forces it changes its shape more or less, and that when this change of shape becomes sufficiently great the body breaks. It has also been found that the amount of change in shape necessary to cause rupture depends on the material of which the body is made. For instance, a piece of vulcanized rubber will stretch about eight times its own length before breaking, while if a piece of steel is stretched until it breaks, the elongation preceding rupture is only from $\frac{1}{10}$ to $\frac{1}{5}$ of its original length.

2. Subject-matter of the strength of materials. Since the assumption of rigidity upon which mechanics is based cannot be extended to natural bodies, mathematical analysis alone is not sufficient to determine the strength of any given structure. A knowledge of the physical properties peculiar to the material of which the structure is made is also essential.

The subject-matter of the strength of materials, therefore, consists of two parts. First, a mathematical theory of the relation between the external forces which act on a body and its resultant change of shape, by means of which the direction and intensity of the forces acting at any point of the body may be calculated; and, second, an

experimental determination of the physical properties, such as strength and elasticity, of the various materials used in construction.

Although it is convenient to divide the subject in this way, it must be understood that the two parts are, in reality, inseparable; for the mathematical discussion involves physical constants which can be found only by experiment, while, on the other hand, experiment alone is powerless to determine the form which should be given to construction members in order to secure efficiency of design with economy of material.

3. Stress, strain, and deformation. Whenever an external force acts on a body it creates a resisting force within the body. This, in fact, is simply another way of stating Newton's third law of motion, that to every action there exists a reaction equal in magnitude and opposite in direction. This internal resistance is due to innumerable small forces of attraction exerted between the molecules of the body, called "molecular forces," or **stresses**. A body subjected to the action of stress is said to be **strained**, and the resulting change in shape is called the **deformation**.

For example, suppose a copper wire 40 in. long supports a weight of 10 lb. and is stretched by this weight so that its length becomes 40.1 in. Then the sum of the stresses acting on any cross section of the wire is 10 lb., and the effect of this stress is to strain the wire until its deformation, or increase in length, is .1 in.

4. Tension, compression, and shear. In order to determine the relation between the stresses at any point in a solid body, only a small portion of the body is considered at a time, say an infinitesimal cube. This small cube is then assumed to act like a rigid body, and the relations between the stresses which act on it are determined by means of the conditions of equilibrium deduced in mechanics.

By the principle of the resolution of forces, the stresses acting on any face of such an elementary cube can be analyzed into two components, one perpendicular to the face of the cube and the other lying in the plane of the face. That component of the stress which is perpendicular to the face of the cube is called the **normal stress**. If the normal stress pulls on the cube, and thus tends to increase its dimensions, it is called **tension**; if it pushes on the cube, and thus tends to decrease its dimensions, it is called **compression**. Tension is indicated by the sign + and compression by the sign −.

That component of the stress which lies in the plane of the face tends to slide this face past the adjoining portion of the body, and for this reason is called the **shear**, since its action resembles that of a pair of scissors or shears.

5. Unit stress. If the total stress acting on any cross section of a body is divided by the area of the cross section, the result is the stress per unit of area, or **unit stress**. In what follows p will be used to denote the unit normal stress and q to denote the unit shear. Thus if a bar 2 in. square is stretched by a force of 800 lb., the unit normal stress is

$$p = + \frac{800 \text{ lb.}}{4 \text{ in.}^2} = + 200 \text{ lb./in.}^2 \, *$$

If a rod is subjected to tension, it is customary to assume that the stress is uniformly distributed over any cross section of the rod. This assumption, however, is only approximately correct; for if two parallel lines are drawn near the center of a rubber test piece, as ab and cd in Fig. 1, A, it is found that when the test piece is subjected to tension these two lines become convex toward one another, as indicated in Fig. 1, B, showing that the tensile stress is greater near the edges of the piece than at the center. In such a case of nonuniform distribution of stress, the smaller the area considered the nearer the unit stress approaches its true value. That is to say, if ΔP is the stress acting on a small area ΔF, then, in the notation of the calculus,

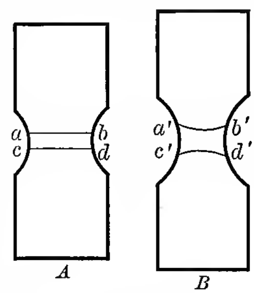


FIG. 1

$$p = \lim \frac{\Delta P}{\Delta F} = \frac{dP}{dF}.$$

Problem 1. A post 1 ft. in diameter supports a load of one ton.† Assuming that the stress is uniformly distributed over any cross section, find the unit normal stress.

Problem 2. A shearing force of 50 lb. is uniformly distributed over an area 4 in. square. Find the unit shear.

* For the sake of brevity and clearness all dimensions in this book will be expressed as above; that is, "lb. per sq. in." will be written "lb./in.²," etc.

† Throughout this book the word "ton" is used to denote the net ton of 2000 lb.

6. Unit deformation. If a bar of length l is subjected to tension or compression, its length is increased or diminished by a certain amount, say Δl. The ratio of this change in length to the original length of the bar is called the **unit deformation**, and will be denoted by s. Thus

$$s = \frac{\Delta l}{l}.$$

In other words, the unit deformation is the elongation or contraction per unit of length, or the percentage of deformation, and s is therefore an abstract number.

Problem 3. A copper wire 100 ft. long and .025 in. in diameter stretches 2.16 in. when pulled by a force of 15 lb. Find the unit elongation.

Problem 4. If the wire in Problem 3 was 250 ft. long, how much would it lengthen under the same pull?

Problem 5. A vertical wooden post 30 ft. long and 8 in. square shortens .00374 in. under a load of half a ton. What is its unit contraction?

7. Strain diagrams. As mentioned in Article 1, experiment has shown that the effect of the action of external forces upon a body is to produce a change in its shape. If the body returns to its original shape when these external forces are removed it is said to be **elastic,** whereas if it remains deformed it is said to be **plastic.**

For instance, the steel hairspring of a watch is an example of an elastic body, for although it is compressed thousands of times daily it returns each time to its original shape when the compressive force is removed. Wood, iron, glass, and ivory are other examples of elastic substances.

As examples of plastic bodies may be taken such substances as putty, lead, and wet clay, for such materials retain any shape into which they may be pressed.

It has been found by experiment that most of the materials used in engineering are almost perfectly elastic, if the forces acting on them are not too large. That is to say, if the external forces do not surpass a certain limit, the permanent deformation, although not zero, is so small as to be negligible. If, however, the external forces gradually increase, there comes a time when the body no longer regains its original form completely upon removal of the stress, but takes a permanent "set" due to plastic deformation. If the external forces increase beyond this point, the permanent (or plastic)

deformation also increases; or, in other words, the tendency of the body to return to its original form grows less and less until rupture occurs.

For example, suppose that a rod of steel or wrought iron is stretched by a tensile force applied at its ends. Then if the unit tensile forces acting on the rod are plotted as ordinates and the corresponding unit elongations of the rod as abscissas, a curve will be obtained, as shown in Fig. 2.*

Consider the curve for wrought iron obtained in this way. For stresses less than a certain amount, indicated by the ordinate at A in

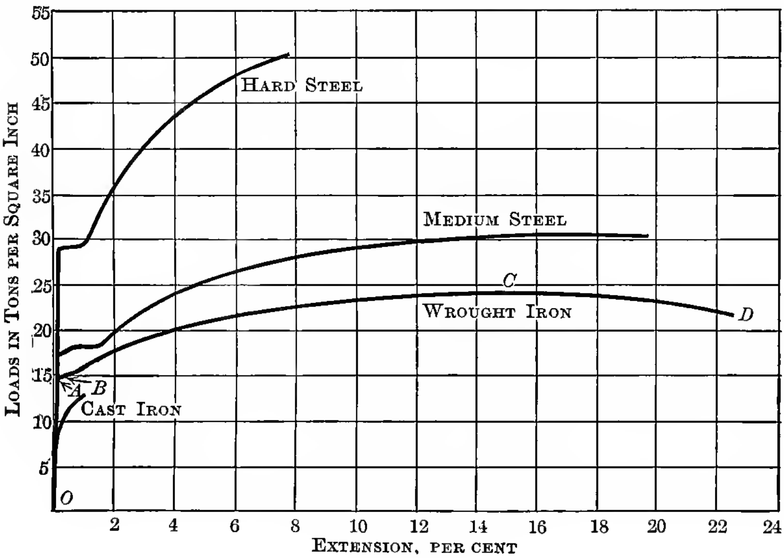


Fig. 2

Fig. 2, the deformation is very slight and is proportional to the stress which produces it, so that this portion OA of the strain diagram is a straight line. For stresses above A the deformation increases more rapidly than the stress which produces it, and consequently the strain diagram becomes curved. When the stress reaches a certain point B the material suddenly yields, the deformation increasing to a marked extent without any increase in the stress. Beyond this point the deformation increases with growing rapidity until rupture is about

* Drawn from data given in the *United States Government Reports on Tests of Metals.*

to take place. At this stage of the experiment, indicated by C on the diagram, the material in the neighborhood of the place where rupture is to occur begins to draw out very rapidly, and in consequence the cross section of the piece diminishes at this point until rupture occurs.

Within the portion OA of the strain diagram the stress is proportional to the deformation produced, and the material may be considered to be perfectly elastic. For this reason the point A, which is the limit of proportionality of stress to deformation, is called the **elastic limit**. The point B, at which the first signs of weakening occur, is called the **yield point**.

In commercial testing the tests are usually conducted so hurriedly that the position of the point A is not noted, and consequently the yield point is often called the elastic limit. The yield point, however, is not the true elastic limit, because plastic deformation begins to be manifested before this point is reached, namely, as soon as the stress passes A.

At C the tangent to the strain curve is horizontal. Therefore the ordinate at this point indicates the maximum stress preceding rupture, which is called the **ultimate strength** of the material.

8. Hooke's law and Young's modulus. The fact that within the elastic limit the deformation of a body is proportional to the stress producing it was discovered in 1678 by Robert Hooke, and is therefore known as **Hooke's law**. It can be stated by saying that the ratio of the unit stress to the unit deformation is a constant; or, expressed as a formula,

$$\frac{p}{s} = E,$$

where E is a constant called the **modulus of elasticity**. E is also called **Young's modulus**, from the name of the first scientist who made any practical use of it.

Since s is an abstract number, E has the same dimensions as p and is therefore expressed in lb./in.^{2} Geometrically E is the slope of the line OA in Fig. 2.

The answers given to the following problems were obtained by using the average values of Young's modulus given in Table I.

Problem 6. A steel cable 500 ft. long and 1 in. in diameter is pulled by a force of 25 tons. How much does it stretch, and what is its unit elongation?

Problem 7. A copper wire 10 ft. long and .04 in. in diameter is tested and found to stretch .289 in. under a pull of 50 lb. What is the value of Young's modulus for copper deduced from this experiment?

Problem 8. A round cast-iron pillar 18 ft. high and 10 in. in diameter supports a load of 12 tons. How much does it shorten, and what is its unit contraction?

Problem 9. A wrought-iron bar 20 ft. long and 1 in. square is stretched .266 in. What is the force acting on it?

9. Poisson's ratio. It has been found by experiment that when a rod is subjected to tension or compression its transverse dimensions are changed as well as its length. For instance, if a round rod is in tension, it increases in length and decreases in diameter, whereas, if the rod is compressed, it decreases in length and increases in diameter. Experiment has also shown that this lateral contraction or expansion is proportional to the change in length of the bar; that is to say, the ratio of the unit lateral deformation to the unit change in length is constant, say $\frac{1}{m}\cdot$ This constant is called **Poisson's ratio,** from the name of its originator.

Poisson's ratio varies somewhat for different materials, but ordinarily lies between $\frac{1}{3}$ and $\frac{1}{4}$. Values of this ratio for a number of materials are given in Table I.

Problem 10. What is the lateral contraction of the bar in Problem 9?

Problem 11. A soft steel cylinder 1 ft. high and 2 in. in diameter bears a weight of 75 tons. How much is its diameter increased?

10. Ultimate strength. From the definition given in Article 7, the **ultimate strength** of a body is the greatest unit stress it can stand without breaking. In calculating the ultimate strength no account is taken of the lateral contraction or expansion of the body, the ultimate strength being defined as the breaking load divided by the original area of a cross section of the piece before strain. The reason for this arbitrary definition of the ultimate strength is that the actual load on any member of an engineering structure usually lies within the elastic limit of the material, and within this limit the change in area of a cross section of the member is so small that it can be neglected.

Tabulated values of the ultimate strength of various materials in tension, compression, and shear are given in Table I.

Problem 12. How great a pull can a copper wire .2 in. in diameter stand without breaking?

Problem 13. How large must a square wrought-iron bar be made to stand a pull of 3000 lb.?

Problem 14. A mild steel plate is $\frac{1}{4}$ in. thick. How wide must it be to stand a pull of 1 ton?

Problem 15. A round wooden post is 6 in. in diameter. How great a load will it bear?

11. Elastic law. Certain substances, notably cast iron, stone, cement, and concrete, do not conform to Hooke's law, in that the deformation is not proportional to the stress which produces it. Consequently, for such substances the strain diagram is nowhere a straight line, but is curved throughout, as shown in the curve for cast iron in Fig. 2. In this case the modulus of elasticity changes from point to point.

In the reports of the U. S. Testing Laboratory at the Watertown Arsenal, the modulus of elasticity is defined as the quotient of the unit stress by the unit deformation minus the permanent set. Thus, if s' denotes the permanent set, this definition makes $E = \dfrac{p}{s - s'}$.

Numerous attempts have been made to determine the equation of the strain curve for various materials which do not conform to Hooke's law, and a corresponding number of formulas, or **elastic laws,** have been proposed. The one which agrees best with experiment is the exponential law, expressed by the formula

$$s = \nu p^{\sigma},$$

where ν and σ are constants determined by experiment. From Bach's experiments the values of ν and σ were found to be such that

$$\text{for cast iron in tension,} \qquad s = \frac{1}{24,268,000} p^{1.0663};$$

$$\text{for cast iron in compression,} \qquad s = \frac{1}{18,469,200} p^{1.0395};$$

the unit stress p being expressed in lb./in.²

$p = (24268000\, s)^{-1.0663}$

$\frac{dp}{ds} = -(24,268,000\, s)^{-2.0663} \cdot 24268000$

However, all such elastic laws are at best merely interpolation formulas which are approximately true within the limits of the experiments from which they were obtained. For this reason it is best to carry out all investigations in the strength of materials under the assumption of Hooke's law, and then modify the results by a factor of safety, as explained in Article 21.

12. Classification of materials. Materials ordinarily used in engineering construction may be divided into three classes, — plastic, supple, and elastic.

Plastic materials are characterized by their inability to resist stress without receiving permanent deformation. Examples of such materials are lead, wet clay, mortar before setting, etc.

Supple bodies are characterized by their lack of stiffness. In other words, supple bodies are capable of undergoing large amounts of elastic deformation without receiving any plastic deformation. In this respect plastic and supple bodies exhibit the two extremes of physical behavior. Examples of supple bodies are rubber, copper, rope, cables, textile fabrics, etc.

Elastic bodies comprise all the hard and rigid substances, such as iron, steel, wood, glass, stone, etc. For such bodies the plastic deformation for any stress within the elastic limit is so small as to be negligible; but when the stress surpasses this limit the plastic deformation becomes measurable and gradually increases until rupture occurs. This permanent deformation is the outward manifestation of a change in the molecular arrangement of the body. For a stress within the elastic limit the forces of attraction between the molecules are sufficiently great to hold the molecules in equilibrium; but when the stress surpasses the elastic limit, the molecular forces can no longer maintain equilibrium and a change in the relation between the molecules of the body takes place, which results in the body taking a permanent set.

Rigid bodies have the character of supple bodies when one of their dimensions is very small as compared with the others. An instance of this is the flexibility of an iron or steel wire whose length is very great as compared with its diameter. Furthermore, rigid bodies behave like plastic bodies when their temperature is raised to a certain point. For example, when iron and steel are heated to a cherry redness they become plastic and acquire the property of uniting by contact.

13. Time effect. It has been found by experiment that elastic deformation is manifested simultaneously with the application of a stress, but that plastic deformation does not appear until much later. Thus if a constant load acts for a considerable time, the deformation

gradually increases; and when the load is removed the return of the body to its original configuration is also gradual. This phenomenon of the deformation lagging behind the stress which produces it is called **hysteresis.** The gradual increase in the deformation under constant stress is also called the **flow** of the material; and the gradual return of the body to its original shape upon removal of the stress is known as **elastic afterwork.** This gradual flow which occurs under constant stress approaches a limit if the stress lies below the elastic limit, but continues up to fracture if the stress is sufficiently great.

14. Fatigue of metals. If a stress lies well within the elastic limit, it can be removed and repeated as often as desired without causing rupture. If, however, a metal is stressed beyond the elastic limit, and this stress is removed and repeated, or alternates between tension and compression, a sufficient number of times, it will eventually cause rupture. This phenomenon is known as the **fatigue of metals**, and has been made the subject of laborious experiment by Wöhler, Bauschinger, and others. The results of their experiments show that the less the range of variation of stress, the greater the number of repetitions or reversals of stress necessary to produce rupture. Among other results Bauschinger found that for cast iron with an ultimate tensile strength of 64,100 lb./in.^{2}, the maximum tensile stress which could be removed and repeated indefinitely without causing rupture was 35,300 lb./in.^{2}; and that the maximum stress which could be alternated indefinitely between tension and compression of equal amounts without causing rupture was 29,100 lb./in.^{2} For other kinds of iron and steel Bauschinger obtained similar results, the limit of reversible stress in each case agreeing closely with the elastic limit. From this we conclude that the elastic limit of a material is much more important than its ultimate strength in determining the stability of an engineering structure of which it forms a part.

The fatigue of metals indicates that dislocation of matter begins to be produced as soon as the elastic limit is passed, and continues under the action of relatively small forces. This is confirmed by the well-known fact that if, as the result of a blow, a fissure or crack is started in a piece of glass or cast iron, this fissure will spread without any apparent cause until the piece breaks in two, the only way

of stopping this tendency to spread being by boring a small hole at either end of the fissure.

The explanation of the above is that for stresses within the elastic limit the temperature of the body is not raised, and consequently all the work of deformation is stored up in the body to be given out again in the form of mechanical energy upon removal of the stress. If, however, the elastic limit is surpassed, the friction of the molecules sliding on each other generates a certain amount of heat, and the energy thus transformed into heat is not available for restoring the body to its original configuration.

15. Hardening effects of overstraining. When such materials as iron and steel are stressed beyond the elastic limit, it is found upon removal of the stress that the effect of this overstrain is a hardening of the material, and that this hardening increases indefinitely with time. For example, if a plate of soft steel is cold punched, the material surrounding the hole is severely strained. After an interval of rest the effects of this overstrain is manifested in a hardening of the material which continues to increase for months. If the plate is subsequently stressed, the inability of the portion overstrained to yield with the rest of the plate causes the stress to be concentrated on these portions, and results in a serious weakening of the plate.

Other practical instances of hardening due to overstrain are found in plates subjected to shearing and planing, armor plates pierced by cannon balls, plates and bars rolled, hammered, or bent when cold, wire cold drawn, etc.

16. Fragility. In the solidification of melted bodies different parts are unequally contracted or expanded. This gives rise to internal stresses, or what is called **latent molecular action**, and puts the body in a state of strain without the application of any external forces. For instance, if a drop of melted glass is allowed to fall into water, the outside of the drop is instantly cooled and consequently contracted, while the inside still remains molten. Since the part within cannot contract while molten, the contraction of the outside causes such large internal stresses that the glass is shattered.

Bodies in which latent molecular action exists have the character of an explosive, in that they are capable of standing a large static stress but are easily broken by a blow, and for this reason they are

called brittle or **fragile**. The explanation of fragility is that the vibrations caused by a blow are reinforced by the latent internal stresses until rupture ensues.

17. Initial internal stress. In certain bodies, such as cast iron, stone, and cement, a state of internal stress may exist without the application of any external force. This initial internal stress may be the result of deformation caused by previously applied loads, or may be occasioned by temperature changes, as mentioned in the preceding article. The first load applied to such bodies gives them a slight permanent deformation, but under subsequent loads their behavior is completely elastic. The first load, in this case, serves to relieve the strain due to initial internal stress, and consequently the behavior of the body under subsequent loads is normal. A body which is free from internal stress is said to be in a "state of ease," a term which is due to Professor Karl Pearson.

18. Annealing. The process of annealing metals consists in heating them to a cherry redness and then allowing them to cool slowly. The effect of this process is to relieve any initial internal stress, or stress due to overstrain, and put the material in a state of ease. Hardening due to overstrain is of frequent occurrence in engineering, and the only certain remedy for it is annealing. If this is impracticable, hardening can be practically avoided by substituting boring for punching, sawing for shearing, etc.

19. Temperature stresses. A property especially characteristic of metals is that of expansion with rise of temperature. The proportion of its length which a bar expands when its temperature is raised one degree is called the **coefficient of linear expansion**, and will be denoted by L. The following table gives the value of L for one degree Fahrenheit for the substances named.

Steel, hard	$L = .0000074$
" soft	$L = .0000061$
Iron, cast	$L = .0000063$
" wrought	$L = .0000068$
Timber	$L = .0000028$
Granite	$L = .0000047$
Sandstone	$L = .0000065$

If a body is fixed to immovable supports so that when the temperature of the body is raised these supports prevent it from expanding,

stresses are produced in the body called **temperature stresses.** Thus suppose a bar of length l is rigidly fastened to immovable supports and its temperature is then raised a certain amount. Let Δl be the amount the bar would naturally lengthen under this rise in temperature if left free to move. Then the stress necessary to produce a shortening of this amount is the temperature stress.

If the temperature of the bar is raised T degrees,

$$\Delta l = LlT,$$

and consequently
$$s = \frac{\Delta l}{l} = LT.$$

Therefore, if p denotes the unit temperature stress,

$$\boldsymbol{p = sE = LTE}.$$

The temperature of metals also has a marked influence upon their ultimate strength. Experiments along this line show that at $-296°$ F. the tensile strengths of iron and steel are about twice as great as at ordinary temperatures.

Problem 16. A wrought-iron bar is 20 ft. long at 32° F. How long will it be at 95° F. ?

Problem 17. A cast-iron pipe 10 ft. long is placed between two heavy walls. What will be the stress in the pipe if the temperature rises 25° ?

Problem 18. Steel railroad rails, each 30 ft. long, are laid at a temperature of 40° F. What space must be left between them in order that their ends shall just meet at 100° F. ?

Problem 19. In the preceding problem, if the rails are laid with their ends in contact, what will be the temperature stress in them at 100° F. ?

20. Effect of length, diameter, and form of cross section. When an external force is first applied to a body the internal stress is distributed uniformly throughout the body and, consequently, all parts are equally deformed.* When the stress surpasses the elastic limit this is no longer true, and certain portions of the body begin to manifest greater deformation than others. For instance, consider a bar of soft steel under tension. As the stress increases from zero to the elastic limit the bar gradually lengthens and its cross section diminishes, all parts being equally affected. When the stress passes beyond the elastic limit the cross section at some particular point of the bar,

* This depends somewhat upon the way the external force is applied.

usually near the center, begins to diminish more rapidly than elsewhere. This contraction of section intensifies the unit stress at this point, and this in turn tends to a still greater reduction of section until finally rupture occurs.

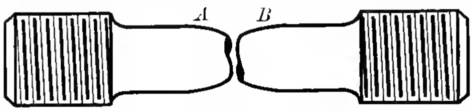


Fig. 3

The appearance of a bar subjected to a test of this kind is represented in Fig. 3. The contracted portion, *AB*, of the bar is called the **region of striction**. The contraction of the section at which rupture occurs is usually considerable; for soft steel its amount is from .4 to .6 of the original area of the bar.

In Article 6 the unit elongation was defined as the ratio of the total elongation to the original length of the bar. It has been found by experiment, however, that the extent of the region of striction depends on the transverse dimensions of the bar and not on its length, the region of striction increasing in extent as the transverse dimensions of the bar increase. Consequently, if two bars are of equivalent cross section but of different lengths, the region of striction will be the same for both, and therefore the unit elongation will appear to be less for the long bar than for the short one. On the other hand, if the two bars are of the same length, but one is thicker than the other, the region of striction will be longer for the thick bar, and therefore the unit elongation of this bar will appear to be greater than for the other.

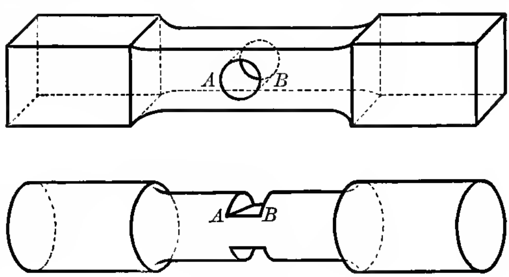


Fig. 4

The form of cross section of test pieces subjected to tensile tests has also an important influence on their elongation and on their ultimate strength. If a sharp change in cross section occurs at any point, nonductile materials, such as cast iron, will break at this section under a smaller unit stress than they could otherwise carry. This is due to a greater intensity of stress at the section where the change in area occurs.

For ductile materials, such as wrought iron and mild steel, the striction extends over a length six or eight times the width of the piece. Consequently, if the test piece has a form similar to one of those represented in Fig. 4, in which the length AB is less than six or eight times the width of the piece, the flow of the metal is restrained and therefore its ultimate strength is raised. This has an important bearing on the strength of riveted plates subjected to tensile strain. It has been experimentally proved that such plates will stand a greater tension than plates of uniform cross section whose sectional area is equal to the sum of the sectional areas between the rivet holes.

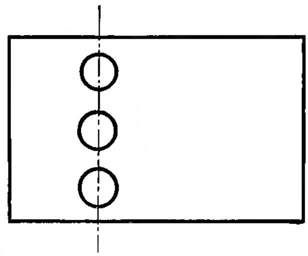


Fig. 5

In Article 10 the ultimate strength was defined as the ratio of the maximum stress to the original sectional area of the bar. It is evident from what precedes, therefore, that the unit elongation and the ultimate strength are not absolute quantities, but depend on the form of the test piece and the conditions of the test. For this reason it is absolutely essential that the results of any test be accompanied by an accurate description of the circumstances under which they were obtained. The elastic limit and modulus of elasticity, on the contrary, have an intrinsic value independent of their method of determination, and therefore more accurately define the elastic properties of any material.

The tensile strength of long rods is affected in a way different from any of the preceding. Since no material is perfectly homogeneous, the longer the rod the greater the chance that a flaw will occur in it somewhere. If, then, by numerous tests of short pieces, it has been determined how much a material lacks of being homogeneous, the strength of a rod of this material of any given length can be calculated by means of the theory of probabilities. Such a theory has been worked out by Professor Chaplin * and verified experimentally.

If one dimension of a body is very small compared with the others, as, for example, in long wires or very thin plates, the body

* *Van Nostrand's Eng. Mag.*, December, 1880; also *Proc. Eng. Club*, Philadelphia, March, 1882.

may be permanently deformed by stresses below the elastic limit. The reason for this is that the smallest dimension of such a body is of the same order of magnitude as the deformation of one of the other dimensions, and consequently Hooke's law does not apply in this case.

21. Factor of safety. In order to assure absolute stability to any structure it is clear from what precedes that the actual stresses occurring in the structure must not exceed the elastic limit of the material used.

For many materials, however, it is very difficult to determine the elastic limit, while for other materials for which the determination is easier, such as iron and steel, the elastic limit is subject to large variations in value, and it is impossible to do more than assign wide limits within which it may be expected to lie. For this reason it is customary to judge the quality of a material by its ultimate strength instead of by its elastic limit, and assume a certain fraction of the ultimate strength as the allowable working stress.

The number which expresses the ratio of the ultimate strength to the working stress is called the **factor of safety**. Thus

$$\textbf{Factor of safety} = \frac{\textbf{ultimate strength}}{\textbf{working stress}}.$$

No general and rational method of determining the factor of safety can be given. For, in the first place, formulas deduced from theoretical considerations rest on the assumption that the material considered is perfectly elastic, homogeneous, and isotropic,—an assumption which is never completely fulfilled. Such formulas give, therefore, only an approximate idea of the state of stress within the body.

Moreover, the forms of construction members assumed for purposes of calculation do not exactly correspond to those actually used; also certain conditions are unforeseen, and therefore unprovided for, such as the sinking of foundations, accidental shocks, etc.

In metal constructions rust is another element which tends to reduce their strength, and in timber constructions the same is true of wet and dry rot. Care is usually taken to prevent rust and decay, but the preservative processes used never perfectly accomplish their object.

Besides these elements of uncertainty every construction is attended by its own peculiar circumstances, such as the duration to be given to it, the gravity of an accident, etc., which requires a special determination of the factor of safety.

For all these reasons it is impossible to definitely fix a factor of safety which will fit all cases, and the only guide that can be given as to its choice is to say that it will lie between certain limits. According to Résal,* the factor of safety for iron, steel, and ductile metals should be 4 or 3, and never less than $2\frac{1}{2}$; for heterogeneous materials, such as cast iron, wood, and stone, the factor of safety should lie between 20 and 10, and never be less than the latter.

Problem 20. In the United States government tests of rifle-barrel steel it was found that for a certain sample the unit tensile stress at the elastic limit was 71,000 lb./in.^{2}, and that the ultimate tensile strength was 118,000 lb./in.^{2} What must the factor of safety be in order to bring the working stress within the elastic limit?

Problem 21. In the United States government tests of concrete cubes made of Atlas cement in the proportions of 1 part of sand to 3 of cement and 6 of broken stone, the ultimate compressive strength of one specimen was 883 lb./in.^{2}, and of another specimen was 3256 lb./in.^{2} If the working stress is determined from the ultimate strength of the first specimen by using a factor of safety of 5, what factor of safety must be used to determine the same working stress from the other specimen?

Problem 22. An elevator cab weighs 3 tons. With a factor of safety of 5, how large must a steel cable be to support the cab? (Use Roebling's tables for wire rope given in Part II.)

22. Work done in producing strain. In constructing the strain diagram, explained in Article 7, the unit stresses were plotted as ordinates and the corresponding unit deformations as abscissas. The autographic apparatus on a testing machine also gives a diagram which represents the strain, but in which the loads are the ordinates and the corresponding total deformations are the abscissas. The two diagrams are similar up to the elastic limit but not beyond this point, for after the elastic limit is passed, the area of cross section begins to change appreciably so that the unit stress is no longer proportional to the load. If, however, the unit stress is obtained by dividing the load by the original area of cross section, without taking into account the lateral deformation, the plotted strain diagram will be similar to the autographic load-deformation diagram.

* Résal, *Résistance des Matériaux*, p. 195.

The load-deformation diagram has a special physical significance, namely that the area under the curve up to any point represents the work done in producing the strain up to that point. In this respect the autographic strain diagram resembles the indicator diagram on a steam engine.

Since the elastic limit marks the limit within which the material may be considered as perfectly elastic, the area under the strain curve up to the elastic limit represents the amount of work which can be stored up in the form of potential energy, and is called the **resilience** of the test piece. Thus, if p denotes the unit stress at the elastic limit and F the area of cross section, the load is Fp; and hence if Δl denotes the total deformation at the elastic limit, the work done up to this point is $\frac{1}{2} pF\Delta l$. From Hooke's law, $\Delta l = \frac{pl}{E}$. Consequently the expression for the resilience becomes $\frac{1}{2}\frac{p^2 lF}{E}$, or, since lF represents the volume V of the test piece, this may be written $\frac{1}{2}\frac{p^2 V}{E}$. The resilience per unit volume, $\frac{1}{2}\frac{p^2}{E}$, is called the **modulus of elastic resilience** of the material.

EXERCISES ON CHAPTER I

Problem 23. A $\frac{5}{8}$-in. wrought-iron bolt failed in the testing machine under a pull of 20,000 lb. Diameter at root of thread = .5039 in.; find its ultimate tensile strength.

Problem 24. Four $\frac{1}{2}$-in. steel cables are used with a block and tackle on the hoist of a crane whose capacity is rated at 6000 lb. What is the factor of safety? (Use Roebling's tables, Part II, for ultimate strength of rope.)

Problem 25. A vertical hydraulic press weighing 100 tons is supported by four $2\frac{1}{2}$-in. round cold-rolled steel rods. Find the factor of safety.

Problem 26. A block and tackle consists of six strands of flexible $\frac{1}{2}$-in. steel cable. What load can be supported with a factor of safety of 5?

Problem 27. A wooden bar 6 ft. long, suspended vertically, is found to lengthen .013 in. under a load of 2100 lb. hung at the end. Find the value of E for this bar.

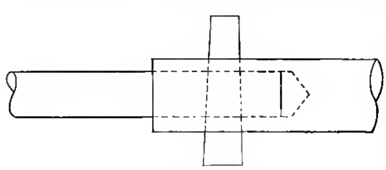


FIG. 6

Problem 28. A copper wire $\frac{1}{4}$ in. in diameter and 500 ft. long is used as a crane trolley. The wire is stretched with a force of 100 lb. when the temperature is 80° F. Find the pull in the wire when the temperature is 0° F., and the factor of safety.

Problem 29. An extended shank is made for a $\frac{17}{32}$-in. drill by boring a $\frac{17}{32}$-in. hole in the end of a 10-in. length of cold-rolled steel, fitting the shank into this and putting a steel taper pin through both (Fig. 6). Standard pins taper $\frac{1}{4}$ in. per foot.

What size pin should be used in order that the strength of the pin against shear may equal the strength of the drill shank in compression around the hole?

Problem 30. The head of a steam cylinder of 12-in. inside diameter is held on by 10 wrought-iron bolts. How tight should these bolts be screwed up in order that the cylinder may be steam tight under a pressure of 180 lb./in.^{2}?

Problem 31. Find the depth of head of a wrought-iron bolt in terms of its diameter in order that the tensile strength of the bolt may equal the shearing strength of the head.

Problem 32. The pendulum rod of a regulator used in an astronomical observatory is made of nickel steel in the proportion of 35.7 per cent nickel to 64.3 per cent steel. The coefficient of expansion of this alloy is approximately $\frac{1}{12}$ that of steel, $\frac{1}{18}$ that of copper, and $\frac{1}{23}$ that of aluminum. This is tempered for several weeks, starting at 180° F. and gradually lowering to the temperature of the room, which eliminates the effect of elastic afterwork.

The rod carries two compensation tubes, A and B, Fig. 7, one of copper and the other of steel, the length of the two together being 10 cm. If the length of the rod is 1 m., find the lengths of the two compensation tubes so that a change in temperature shall not affect the length of the pendulum.

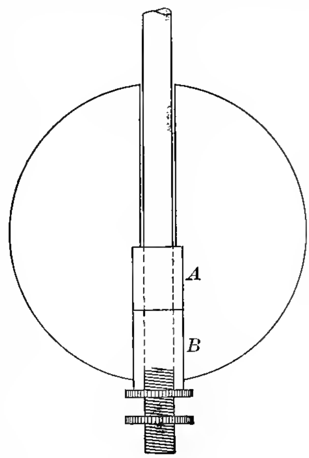


FIG. 7

Problem 33. Refer to the *Watertown Arsenal Reports* (*United States Government Reports on Tests of Metals*), and from the experimental results there tabulated draw typical strain diagrams for mild steel, wrought iron, cast iron, and timber, and compute E in each case.

Problem 34. A steel wire $\frac{1}{4}$ in. in diameter and a brass wire $\frac{1}{2}$ in. in diameter jointly support a load of 1200 lb. If the wires were of the same length when the load was applied, find the proportion of the load carried by each.

Problem 35. An engine cylinder is 10 in. inside diameter and carries a steam pressure of 80 lb./in.^{2} Find the number and size of the bolts required for the cylinder head for a working stress in the bolts of 2000 lb./in.^{2}

Problem 36. Find the required diameter for a short piston rod of hard steel for a piston 20 in. in diameter and steam pressure of 125 lb./in.^{2} Use factor of safety of 8.

Problem 37. A rivet $\frac{1}{2}$ in. in diameter connects two wrought-iron plates each $\frac{3}{8}$ in. thick. Compare the shearing strength of the rivet with the crushing strength of the plates around the rivet hole.

CHAPTER II

FUNDAMENTAL RELATIONS BETWEEN STRESS AND DEFORMATION

23. Relations between the stress components. In order to determine the relation between the stresses and deformations within an elastic body, it is necessary to make certain assumptions as to the nature of the body and the manner in which the external forces are applied to it.

The first assumption to be made is that the material of which the body is composed is homogeneous; that is to say, that the elastic properties of any two samples taken from different parts of the body are exactly alike. If, moreover, the surface of the body is continuous and the external forces are distributed continuously over this surface, or, in other words, if there are no cracks or other sudden changes of section in the body, and the external forces are distributed over a considerable bearing surface, it follows, in consequence of the above assumptions, that the deformation at any point of the body is a continuous function of the coördinates of that point. In other words, under the above assumptions the deformation at any point of the body differs only infinitesimally from the deformation at a neighboring point.

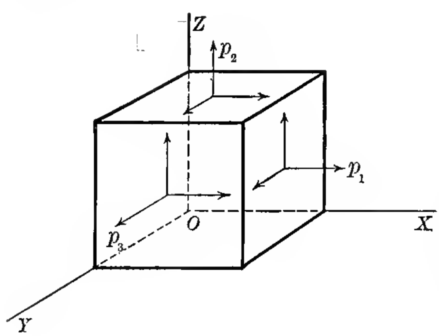


Fig. 8

Since, by Hooke's law, the stress is proportional to the deformation, it follows that the stress is also distributed continuously throughout the body, — that is, that the stress at any point of the

body differs only infinitesimally from the stress at a neighboring point. This is called the **law of continuity.**

Now consider an infinitesimal cube cut out of an elastic body which is subject to the above assumptions, and let the coördinate axes be taken along three adjacent edges of the cube, as shown in Fig. 8. Then, from the law of continuity, the resultant of the normal stresses acting on any face of this cube is equal to their sum and is applied at the center of gravity of the face. Consequently, these resultants must all lie in one or other of the three diametral planes drawn through the center of the cube parallel to the coördinate planes. The stresses lying in any one of these planes, say the diametrical plane parallel to ZOX, will then be as represented in Fig. 9.

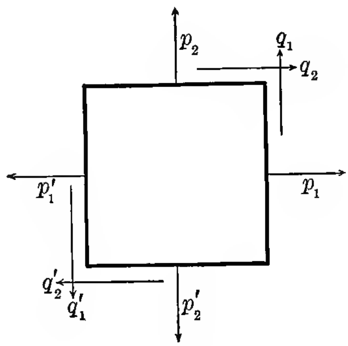


FIG. 9

Since the resultant normal stresses on opposite faces of the cube approach equality as the faces of the cube approach coincidence, we may write

$$p_1 = p_1' \quad \text{and} \quad p_2 = p_2'.$$

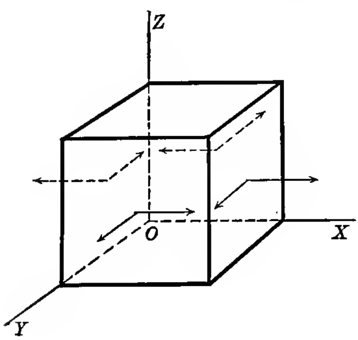


FIG. 10

For equilibrium against rotation the four shearing stresses must also be of equal intensity, and therefore

$$q_1 = q_1' = q_2 = q_2'.$$

By considering the other two diametral planes similar relations between the normal and shearing stresses can be established. Consequently, *the shearing stresses at any point in an elastic body in planes mutually at right angles are of equal intensity in each of these planes.*

24. Planar strain. If no stress occurs on one pair of opposite faces of the cube, the resultant stresses on the other faces all lie in one of the diametral planes. This is called the **planar condition of strain.**

Suppose the Z-axis is drawn in the direction in which no stress occurs, as shown in Fig. 10. Then the stresses all lie in the plane parallel to XOY, and the relation between them is as represented in Fig. 9 of the preceding article.

25. Stress in different directions. As an application of planar stress, consider a triangular prism on which no stress occurs in the direction of its length. Let the Z-axis be drawn in the direction in which no stress occurs, and let α denote the angle which the

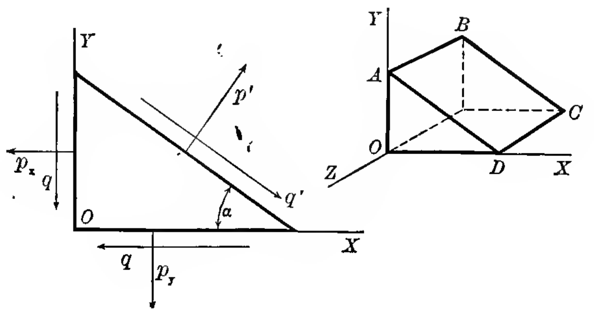


FIG. 11

inclined face of the prism makes with the horizontal, as shown in Fig. 11. Then if dF denotes the area of the inclined face $ABCD$, and p', q' denote the normal and shearing stresses on this face respectively, p' and q' can be expressed in terms of p_x, p_y, and q by means of the conditions of equilibrium. Thus, from Σ hor. comps. $= 0$,

$$p'dF\sin\alpha + q'dF\cos\alpha - p_xdF\sin\alpha - qdF\cos\alpha = 0.$$

Similarly, from Σ vert. comps. $= 0$,

$$p'dF\cos\alpha - q'dF\sin\alpha - p_ydF\cos\alpha - qdF\sin\alpha = 0.$$

Dividing by dF, these equations become

$$(1)\qquad \begin{cases} p'\sin\alpha + q'\cos\alpha - p_x\sin\alpha - q\cos\alpha = 0, \\ p'\cos\alpha - q'\sin\alpha - p_y\cos\alpha - q\sin\alpha = 0. \end{cases}$$

Eliminating q',

$$p' = p_x \sin^2\alpha + p_y \cos^2\alpha + 2q \sin\alpha \cos\alpha.$$

From trigonometry,

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \quad 2\sin\alpha\cos\alpha = \sin 2\alpha.$$

Therefore, by substituting these values,

(2) $$p' = \frac{p_x + p_y}{2} + \frac{p_y - p_x}{2}\cos 2\alpha + q \sin 2\alpha.$$

Similarly, by eliminating p' from equations (1),

(3) $$q' = \frac{p_x - p_y}{2}\sin 2\alpha + q\cos 2\alpha.$$

Problem 38. At a certain point in a vertical cross section of a beam the unit normal stress is 300 lb./in.^{2}, and the unit shear is 100 lb./in.^{2} Find the normal stress and the shear at this point in a plane inclined at 30° to the horizontal.

Solution. Suppose a small cube cut out of the beam at the point N (Fig. 12). Then, by the theorem in Article 23, there will also be a unit shear of intensity q on the top and bottom faces of the cube. In the present case, therefore, $p_x = 300$ lb./in.^{2}, $p_y = 0$, and $q = 100$ lb./in.^{2} Substituting these values in equations (2) and (3), and putting $\alpha = 30°$, the unit normal stress and unit shear on a plane through N inclined at 30° to the horizontal are $p' = 161.5$ lb./in.^{2}, $q' = 179.8$ lb./in.^{2}

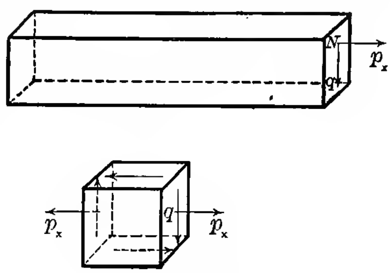


FIG. 12

26. Maximum normal stress. The condition that p' shall be a maximum or a minimum is that $\dfrac{dp'}{d\alpha} = 0$. Applying this condition to equation (2),

(4) $$0 = -\frac{p_y - p_x}{2}\, 2\sin 2\alpha + 2q\cos 2\alpha;$$

whence

(5) $$\tan 2\alpha = \frac{2q}{p_y - p_x},$$

and consequently

(6) $$\alpha = \frac{1}{2}\tan^{-1}\frac{2q}{p_y - p_x} + \frac{\lambda\pi}{2},$$

where λ is zero or an arbitrary integer, either positive or negative. Equation (6) gives the angles which the planes containing the maximum and minimum normal stresses make with the horizontal.

From equation (5),

$$\sin 2\alpha = \pm \frac{2q}{\sqrt{4q^2 + (p_x - p_y)^2}}, \qquad \cos 2\alpha = \pm \frac{p_y - p_x}{\sqrt{4q^2 + (p_x - p_y)^2}}.$$

Substituting these values of $\sin 2\alpha$ and $\cos 2\alpha$ in equation (2), the maximum and minimum values of the normal stress are found to be

$$(7) \qquad p'_{\substack{\max \\ \min}} = \frac{p_x + p_y}{2} \pm \frac{1}{2}\sqrt{4q^2 + (p_x - p_y)^2}.$$

27. Principal stresses. Since λ in equation (6) is an integer, the two values of α given by this equation differ by 90°, and, consequently, the planes containing the maximum and minimum normal stresses are at right angles. The maximum and minimum normal stresses are called **principal stresses**, and the directions in which they act, **principal directions**.

From equation (3), the right member of equation (4) is equal to $2q'$. But since equation (4) is the condition for a maximum or minimum value of the normal stress, it is evident that the normal stress is greatest or least when the shear is zero.

The results of this article can therefore be summed up in the following theorem.

Through each point of a body subjected to planar strain there are two principal directions at right angles, in each of which the shear is zero.

Problem 39. Find the principal stresses and the principal directions at a point in a vertical cross section of a beam at which the unit normal stress is 400 lb./in.^{2} and the unit shear is 250 lb./in.^{2}

Solution. In this problem $p_x = 400$ lb./in.^{2}, $p_y = 0$, and $q = 250$ lb./in.^{2} Therefore, from equation (6),

$$\alpha = \frac{1}{2}\tan^{-1}\frac{5}{-4} + \frac{\lambda\pi}{2} = -25° \, 40.2', \text{ or } +64° \, 19.8';$$

and from equation (7),

$$p'_{\max} = 520 \text{ lb./in.}^2, \qquad p'_{\min} = -120 \text{ lb./in.}^2$$

28. Maximum shear. The condition that q' shall be a maximum or a minimum is that $\frac{dq'}{d\alpha} = 0$. Applying this condition to equation (3),

$$0 = \frac{p_x - p_y}{2} 2 \cos 2\alpha - 2q \sin 2\alpha;$$

whence

$$\tan 2\alpha = \frac{p_x - p_y}{2q}. \tag{8}$$

By comparing equations (5) and (8) it is evident that $\tan 2\alpha$, from (8), equals $-\cot 2\alpha$, from (5). Therefore the values of 2α obtained from these equations differ by 90°, and hence the values of α differ by 45°. Therefore *the planes of maximum and minimum shear are inclined at 45° to the planes of maximum and minimum normal stress.*

From equation (8),

$$\sin 2\alpha = \pm \frac{p_x - p_y}{\sqrt{4q^2 + (p_x - p_y)^2}}, \quad \cos 2\alpha = \pm \frac{2q}{\sqrt{4q^2 + (p_x - p_y)^2}}.$$

Substituting these values of $\sin 2\alpha$ and $\cos 2\alpha$ in equation (3), the maximum and minimum values of the shear are found to be

$$q'_{\substack{\max \\ \min}} = \pm \tfrac{1}{2}\sqrt{4q^2 + (p_x - p_y)^2}. \tag{9}$$

It is to be noticed that the maximum and minimum values of the shear given by equation (9) are equal in absolute amount and differ only in sign, which agrees with the theorem stated in Article 23.

Problem 40. Find the maximum and minimum values of the shear in Problem 39, and their directions.

29. Linear strain. If a body is strained in only one direction, the strain is said to be **linear**. For instance, a vertical post supporting a weight, or a rod under tension, is subjected to linear strain. The unit normal stress and unit shear acting on any inclined section of a body strained in this way can be obtained by supposing the axes of coördinates drawn in the principal directions and putting $q = 0$ and $p_y = 0$ in equations (2) and (3). These values can also be derived independently, as follows.

Consider an elementary triangular prism, and let the axis of X be drawn in the direction of the linear strain. The stresses acting on the prism will then be as shown in Fig. 13. Let dF denote the area of the inclined face. Then the area of the vertical face is $dF \sin \alpha$. Resolving p_x into components parallel to p' and q' respectively, the conditions of equilibrium are

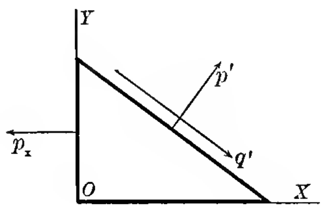


FIG. 13

$$p_x \sin \alpha (dF \sin \alpha) = p' dF,$$
$$p_x \cos \alpha (dF \sin \alpha) = q' dF;$$

or, dividing by dF,

$$(10) \qquad p' = p_x \sin^2 \alpha, \qquad q' = \frac{p_x}{2} \sin 2\alpha.$$

From the condition $\frac{dq'}{d\alpha} = 0$, it is found that the maximum shear occurs when $\alpha = 45°$, and its value is

$$q'_{\max} = \frac{p_x}{2}.$$

For $\alpha = 0°$ or $90°$, $q' = 0$. Consequently, there is no shear in planes parallel or perpendicular to the direction of the linear strain.

Problem 41. A wrought-iron bar 4 in. wide and $\frac{5}{8}$ in. thick is subjected to a pull of 10 tons. What is the unit shear and unit normal stress on a plane inclined at 30° to the axis of the strain? Also what is the maximum unit shear in the bar?

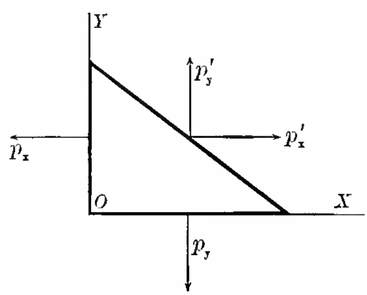


FIG. 14

30. Stress ellipse. Suppose that an elementary triangular prism is cut out of a body subjected to planar strain, so that two sides of the prism coincide with the principal directions. Then, by Article 27, the shears in these two sides are zero. Now let the axes of coördinates be drawn in the principal directions, and resolve the stress acting on the inclined face of the prism into

components parallel to the axes instead of into normal and shearing stresses as heretofore. Then, from Fig. 14, if dF denotes the area of the inclined face, the conditions of equilibrium are

$$p'_x dF = p_x dF \sin\alpha,$$
$$p'_y dF = p_y dF \cos\alpha;$$

whence $$\frac{p'_x}{p_x} = \sin\alpha, \qquad \frac{p'_y}{p_y} = \cos\alpha.$$

Squaring and adding, $$\frac{p'^2_x}{p^2_x} + \frac{p'^2_y}{p^2_y} = 1,$$

which is the equation of an ellipse with semi-axes p_x and p_y, the coördinates of any point on the ellipse being p'_x and p'_y. Consequently, if the stress acting on the inclined face of the prism is calculated for all values of α, and these stresses are represented in magnitude and direction by lines radiating from a common center, the locus of the ends of these lines will be an ellipse called the **stress ellipse** (Fig. 15).

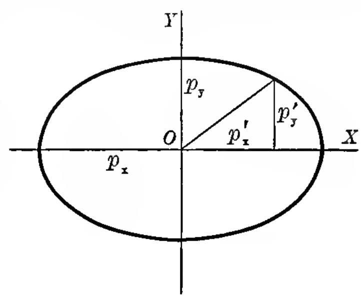


FIG. 15

31. Simple shear. If a body is compressed in one direction and equally elongated in a direction at right angles to the first, the strain is planar. In this case, if the axes are drawn in the principal directions, $q = 0$, $p_x = - p_y$, and the stress ellipse becomes the circle $p'^2_x + p'^2_y = p^2_x$.

Moreover, the normal stress in the planes of maximum or minimum shear is zero; for by substituting in equation (2) the values of $\sin 2\alpha$ and $\cos 2\alpha$ obtained from equation (8), the normal stress in the planes of maximum or minimum shear is found to be $\frac{p_x + p_y}{2}$, and this is zero since $p_x = - p_y$.

Substituting $q = 0$ and $p_x = - p_y$ in equation (9), Article 28, the maximum or minimum value of the shear in the present case is

$$q'_{\substack{\max \\ \min}} = \pm p_x;$$

that is to say, the intensity of the shear in the planes of zero normal stress is equal to the maximum value of the normal stress.

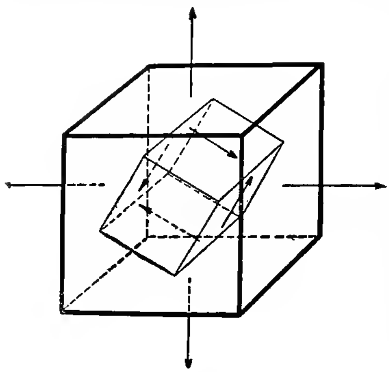


Fig. 16

To give a geometrical representation of the conditions of the problem, suppose a small cube cut out of the body with its faces inclined at 45° to the principal directions. Then the only stresses acting on the inclined faces of this cube are shears equal in amount to the principal stresses. The strain in this case is called **simple shear.**

Conversely, if a small cube is subjected to simple shear, as indicated in Fig. 17, tensile stresses equal in amount to this shear occur in the diagonal plane AC of the cube, and compressive stresses of like amount in the diagonal plane BD.

Problem 42. The steel propeller shaft of a steamship is subjected to a shearing stress of 10,000 lb./in.^{2} Find the maximum tensile stress in the shaft.

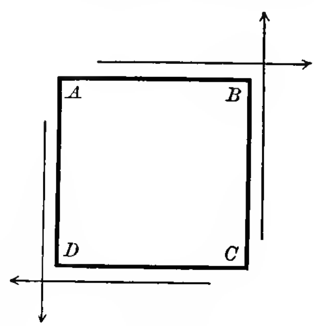


Fig. 17

32. Coefficient of expansion. Consider an infinitesimal prism of dimensions dx, dy, dz, and suppose that under strain these dimensions become $dx + s_x dx$, $dy + s_y dy$, $dz + s_z dz$, where s_x, s_y, s_z are the unit deformations in the directions of the edges of the prism. Then the volume of the prism becomes

$$V + dV = (dx + s_x dx)(dy + s_y dy)(dz + s_z dz),$$

or, neglecting infinitesimals of an order higher than the first,

$$V + dV = (1 + s_x + s_y + s_z)\, dxdydz.$$

Let $K = s_x + s_y + s_z$. Then the change in the volume of the prism due to the strain is

$$dV = Kdxdydz.$$

For this reason K is called the **coefficient of cubical expansion** (or contraction) of the body.

From this definition it is evident that for temperature stresses the coefficient of cubical expansion is three times the coefficient of linear expansion.

From Article 9, for linear tensile strain,

$$s_y = s_z = -\frac{s_x}{m}.$$

Consequently, in this case,

$$K = s_x - \frac{s_x}{m} - \frac{s_x}{m} = \frac{m-2}{m} s_x = \frac{m-2}{m} \cdot \frac{p_x}{E}.$$

Since the prism is certainly not decreased in volume by a tensile strain, K cannot be negative and therefore $m - 2 \geqq 0$, or $m \geqq 2$. If $m = 2$, $K = 0$, which means that the body is incompressible. Therefore 2 is the lower limit of Poisson's constant.

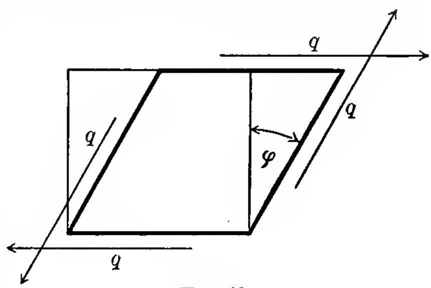


FIG. 18

33. Modulus of elasticity of shear. In an elementary prism subjected to simple shear an angular deformation occurs, as shown in Fig. 18. Let the angle of deformation ϕ be expressed in circular measure. Then, for materials which conform to Hooke's law,

$$\frac{q}{\phi} = G,$$

where G is a constant called the **modulus of elasticity of shear**, or **modulus of rigidity**. Since the angle ϕ, expressed in circular measure, is an abstract number, G must have the dimensions of q, and can therefore be expressed in lb./in.^{2}, as in the case of Young's modulus.

Tabulated values of the modulus of elasticity of shear and ultimate shearing strength for various substances are given in Table I.

Problem 43. A $\frac{3}{4}$-in. wrought-iron bolt has a diameter of .62 in. at base of thread, with a nut $\frac{3}{4}$ in. thick. What force acting on the nut will strip the thread off the bolt?

Problem 44. What force will pull the head off the bolt in Problem 43, if the head is of the same thickness as the nut?

Problem 45. A $\frac{3}{4}$-in. rivet connects two plates which transmit a tension of 2500 lb. Assuming that the shear is uniformly distributed over the cross section of the rivet, find the unit shear on the rivet.

Problem 46. An eyebar is designed to carry a load of 15 tons. What must be the size of the pin to be safe against shear?

NOTE. Consider the pin in double shear, and assume that this shear is uniformly distributed over the cross section of the pin.

34. Relation between the elastic constants. Suppose a cube is subjected to compressive stress on one pair of opposite faces and tensile stress on another pair of opposite faces. Then, if the axes of X and Y are drawn in the direction of the strain, $p_x = -p_y$; and the strain is one of simple shear, as explained in Article 31.

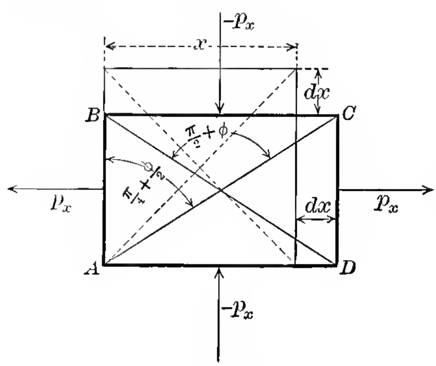


FIG. 19

Let x denote the length of an edge of the cube before strain. Under the strain the cube becomes a parallelopiped, its increase in length in the direction of the X-axis, due to the tensile stress p_x, being $\frac{xp_x}{E}$; and its increase in length in this direction, due to the compressive stress $-p_x$, being $\frac{xp_x}{mE}$.*

Therefore, if dx denotes the total increase in length in the direction of the X-axis,

$$dx = \frac{xp_x}{E} + \frac{xp_x}{mE},$$

or, since $p_x = q$,

$$dx = \frac{m+1}{mE}\,xq.$$

By reason of the strain the angle between the diagonals is increased by an amount ϕ, and therefore the angle between a diagonal and a side is increased by $\frac{\phi}{2}$. From the right triangle ABC (Fig. 19),

* This assumes that the modulus of elasticity is the same for tension as for compression.

$$\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\frac{x+dx}{x-dx}.$$

From trigonometry,

$$\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}}.$$

Since ϕ is assumed to be very small, $\tan\frac{\phi}{2}=\frac{\phi}{2}$, approximately, and therefore

$$\frac{1+\frac{\phi}{2}}{1-\frac{\phi}{2}}=\frac{x+dx}{x-dx};$$

whence $$\phi=\frac{2\,dx}{x}=\frac{2(m+1)}{mE}q.$$

By definition, $G=\frac{q}{\phi}$. Therefore

$$G=\frac{m}{2(m+1)}E, \tag{11}$$

which expresses the relation between the elastic constants G, E, and m.

Problem 47. From the values of G and E, given in Article 22, determine the value of m for cast iron.

35. Measure of strain. In general, the unit deformation s is taken as the measure of a strain. The calculation of s, however, involves a knowledge of the modulus of elasticity E, and for many materials the latter is difficult to determine. To obviate this difficulty, any given strain may be compared with a linear strain which is produced by a unit stress equal to the maximum allowable unit stress. The stress which would produce this linear strain is called the **equivalent stress.**

To illustrate the application of this method, consider a planar strain in which p_1 and p_2 denote the principal stresses and s_1, s_2 the corresponding unit deformations. Then, by Hooke's law, the

stress p_1 acting alone would produce a unit deformation in the direction in which it acts of amount $s_1 = \frac{p_1}{E}$, and also a lateral uuit deformation of $\frac{1}{m}$th this amount, namely $\frac{s_1}{m}$ or $\frac{p_1}{mE}$. Similarly, the stress p_2 acting alone would produce a unit deformation in its own direction of amount $s_2 = \frac{p_2}{E}$, and a deformation at right angles of amount $\frac{s_2}{m}$ or $\frac{p_2}{mE}$. Hence the total deformation in the direction in which p_1 acts, say s_x, is

(12)
$$s_x = \frac{p_1}{E} \pm \frac{p_2}{mE},$$

and similarly the total deformation in the direction in which p_2 acts is

(13)
$$s_y = \frac{p_2}{E} \pm \frac{p_1}{mE}.$$

Now let p_e denote the linear stress which, acting alone, would produce the same unit deformation s_x or s_y; that is to say, p_e is the equivalent linear stress which would have the same effect so far as deformation is concerned as the combined effect of p_1 aud p_2. Then $s_x = \frac{p_e}{E}\left(\text{or } s_y = \frac{p_e}{E}\right)$, and equating these values of s_x and s_y to those given by equations (**12**) and (**13**) above, we have

(14)
$$p_e = p_1 \pm \frac{1}{m} p_2 \quad \text{or} \quad p_e = p_2 \pm \frac{1}{m} p_1.$$

The value of the equivalent stress can thus be calculated directly from the two principal stresses. In order that the strain be safe, the greater of the two values of p_e found from equation (**14**) must not exceed the maximum allowable unit stress.

In the case of simple shear (Article 31) the principal stresses are equal in amount to the shear but of opposite sign; that is,

$$p_1 = +q, \quad p_2 = -q.$$

Therefore, inserting these values in equation (**14**) we have in this case

$$p_e = p_1 - \frac{1}{m} p_2 = q + \frac{1}{m} q = \frac{m+1}{m} q$$

or
$$q = \frac{m}{m+1} p_e.$$

If, then, the working stress in tension or compression is substituted for p_e, the allowable shear is given by this relation.

Problem 48. Find the value of the equivalent stress in Problem 39, and compare it with the principal stresses.

36. Combined bending and torsion. One of the most important applications of the preceding paragraph is to the calculation of the equivalent stress in a beam subjected simultaneously to bending and torsion.

Let the axis of X be drawn in the direction of the axis of the beam. Then on any cross section of the beam there will be a normal stress p_x due to bending, and a shearing stress q due to torsion, while the stress between adjacent longitudinal fibers is zero; that is, $p_y = 0$. Therefore, from equation (7), the principal stresses are

$$p_1 = \tfrac{1}{2}(p_x + \sqrt{4q^2 + p_x^2}), \quad p_2 = \tfrac{1}{2}(p_x - \sqrt{4q^2 + p_x^2}).$$

Consequently, from equation (14), the equivalent stress is

$$p_e = \frac{m-1}{2m} p_x \pm \frac{m+1}{2m} \sqrt{4q^2 + p_x^2}. \tag{15}$$

The sign between the terms depends on which of the two values for p_e in equation (14) is chosen. Evidently that sign should be chosen which will give the most unfavorable value of p_e. Thus on the tension side of a shaft subjected to combined bending and torsion the positive sign should be chosen, and on the compression side the negative sign.

If $m = 3\frac{1}{3}$, which is the best approximate value to use in general, equation (15) becomes

$$p_e = .35\, p_x \pm .64 \sqrt{4q^2 + p_x^2}$$

Many engineers, however, are accustomed to assume .25 for Poisson's ratio, making $m = 4$. The reason for using this value is probably that the modulus of rigidity G for most materials is roughly equal to $.4E$; that is,

$$G = .4E,$$

which by equation (**11**) is equivalent to assuming $m = 4$. For this value of m equation (**15**) becomes

$$p_e = \tfrac{3}{8}\, p_x \pm \tfrac{5}{8}\sqrt{4\, q^2 + p_x^2}.$$

Problem 49. A round steel shaft used for transmitting power bears a transverse load. At the most dangerous section the normal stress due to bending is 5000 lb./in.², and the shear due to torsion is 8000 lb./in.² Calculate the intensity of the equivalent stress.

EXERCISES ON CHAPTER II

Problem 50. In a boiler plate the tensile stress in the direction of the axis of the shell is 2 tons per square inch, and the hoop stress is 4 tons per square inch. Calculate the equivalent linear tensile stress.

Problem 51. At a point in strained material the principal stresses are 0, 9000 lb./in.² tensile, and 5000 lb./in.² compressive. Find the intensity and direction of the resultant stress on a plane inclined 45° to the axis of the tensile stress and perpendicular to the plane which has no stress.

Problem 52. At a point in the cross section of a girder there is a compressive stress of 5 tons/in.² normal to the cross section, and a shearing stress of 3 tons/in.² in the plane of the section. Find the directions and amounts of the principal stresses.

Problem 53. At a certain point in a shaft there is a shearing stress of 5000 lb./in.² in the plane of the cross section, and a tensile stress of 3000 lb./in.² parallel to the axis of the shaft. Find the direction and intensity of the maximum shear.

Problem 54. Solve Problem 51 graphically by drawing the stress ellipse to scale and scaling off the required stress.

Problem 55. In a shaft used for transmitting power the maximum shearing stress, arising from torsional strain, is 5000 lb./in.² Find the normal, or bending, stress it can also carry if the working stress is limited to 10,000 lb./in.² for tension or compression, and to 8000 lb./in.² for shear.

CHAPTER III

ANALYSIS OF STRESS IN BEAMS

37. System of equivalent forces. The theory of beams deals, in general, with the stresses produced in a prismatic body by a set of external forces in static equilibrium. Ordinarily these forces all lie in one plane; in this case it is proved in mechanics that they can be replaced by a single force acting at any given point in this plane, and a moment. To balance this equivalent system of external forces, the stresses acting on any cross section of the beam must also consist of a single force and a moment, the point of application of this single force being conveniently chosen as the center of gravity of the cross section.

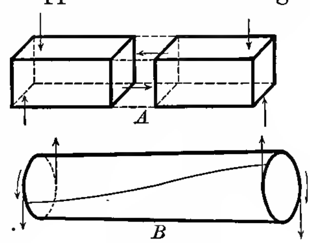


FIG. 20

The following special cases are of frequent occurrence.

If the moment is zero and the single force through the center of gravity of a cross section acts in the direction of the axis of the beam, the strain is simple **tension** or **compression**; if it is perpendicular to the axis of the beam, the strain is simple **shear.**

If the single force is zero and the plane of the moment passes through the axis of the beam, **pure bending strain** occurs; if the single force is zero and the plane of the moment is perpendicular to the axis of the beam, a twisting strain called **torsion** is produced. These two cases are illustrated in Fig. 20, A and B.

If the plane of the moment forms an arbitrary angle with the axis of the beam, the moment can be resolved into two components whose planes are parallel and perpendicular respectively to the axis of the beam. In this case the strain consists of combined bending and torsion.

If the single force through the center of gravity is inclined to the axis of the beam, it can be resolved into two components,—one in the

direction of the axis, called the *axial loading*, and the other perpendicular to the axis, called the *shear*.

38. Common theory of flexure. In the majority of practical cases of flexure (or bending) of beams, the external forces acting on the beam all lie in one plane through its axis and are perpendicular to this axis. The single force through the center of gravity of any cross section is then perpendicular to the axis of the beam, and the plane of the moment passes through this axis. The theory based on the assumption of this condition of strain is called the **common theory of flexure.***

39. Bernoulli's assumption. In order to obtain a starting point for the analysis of stress in beams, the arbitrary assumption is made that *a cross section of the beam which was plane before flexure remains plane after flexure.* This assumption was first made by Bernoulli, and since his time has formed the basis for all investigations in the theory of beams.†

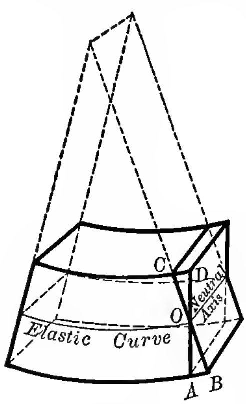


FIG. 21

40. Curvature due to bending moment. The effect of the external moment on a beam originally straight is to cause its axis to become bent into a curve, called the **elastic curve.** Since, by Bernoulli's assumption, any cross section of the beam remains identical with itself during deformation, any two consecutive cross sections of the beam which were perpendicular to its axis before flexure will remain perpendicular to it after flexure, and will therefore intersect in a center of curvature of the elastic curve, as shown in Fig. 21.

The fibers of the beam between these two cross sections were originally of the same length. After flexure, however, it will be found that the fibers on the convex side have been lengthened by a certain amount AB, while those on the concave side have been shortened by an amount

* The common theory of flexure also includes the following assumptions: (1) the assumption that Hooke's law is true (Arts. 8 and 11); (2) the assumption that plane sections remain plane (Art. 39); (3) the neglect of vertical shear deformation (Arts. 68 and 69); (4) the assumption that dl is equal to dx; (5) the assumption that the compressive modulus is equal to the tensile modulus of elasticity; (6) the neglect of conjugate effect from the transverse compression (Art. 9).

† St. Venant has shown that Bernoulli's assumption is rigorously true only for certain forms of cross section. For materials which conform to Hooke's law, however, it is sufficiently exact to assure results approximately correct.

*CD.** Between these two there must lie a strip of fibers which are neither lengthened nor shortened. The horizontal line in which this strip intersects any cross section is called the **neutral axis** of the section.

41. Consequence of Bernoulli's assumption. From Fig. 21 it is evident that, as a consequence of Bernoulli's assumption, the lengthening or shortening of any longitudinal fiber is proportional to its distance from the neutral axis. But, by Hooke's law, the stress is proportional to the deformation produced. Therefore the stress on any longitudinal fiber is likewise proportional to its distance from the neutral axis. Navier was the first to deduce this result from Bernoulli's assumption.

If, then, the stresses are plotted for every point of a vertical strip MN (Fig. 22), their ends will all lie in a straight line, and consequently this distribution of stress is called the **straight-line law.**

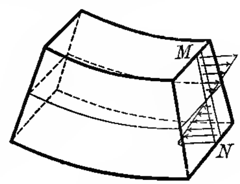


FIG. 22

42. Result of straight-line law. In rectangular coördinates let the axis of Z coincide with the neutral axis, and the axis of Y be perpendicular to it and in the plane of the cross section. Then if the normal stress at the distance y from the neutral axis is denoted by p, and that at a distance y_0 is denoted by p_0, from the straight-line law,

$$\frac{p}{p_0} = \frac{y}{y_0}. \tag{16}$$

Since in order to equilibrate the external bending moment the normal stresses must also form a moment, the sum of the compressive stresses must equal the sum of the tensile stresses. Therefore, since the tensile and compressive stresses are of opposite sign, the algebraic sum of all the normal stresses acting on the section must be zero, that is to say, $\int pdF = 0$, where dF is the infinitesimal area on which p acts. Inserting the value of p from (16),

* This can be shown experimentally by placing two thin steel strips in longitudinal grooves in a wooden beam, one on the upper side and the other on the lower side, so that the strips are free to slide longitudinally but are otherwise fixed. If the strips are of the same length as the beam before bending, it will be found that after bending the upper strip projects beyond the ends of the beam, while the lower strip does not reach the ends. Experiments of this kind have been made by Morin and Tresca. See Unwin, *The Testing of Materials of Construction*, p. 36.

$$\int p dF = \frac{p_0}{y_0}\int y dF = 0,$$

and therefore

$$\int y dF = 0.$$

But the distance of the center of gravity of the section from the axis of Z (or neutral axis) is given by

$$\bar{y} = \frac{\int y dF}{\int dF}.$$

Therefore, since $\int y d F = 0$, $\bar{y}$ must be zero, and consequently *the neutral axis passes through the center of gravity of the section.*

43. Moment of inertia. For equilibrium, the moment of the normal stresses acting on any cross section must equal the moment of the external forces at this section. Therefore, if M denotes the moment of the external forces, or **external bending moment**, as it is called,

$$\int p y dF = M,$$

or, from **(16)**,

$$\frac{p_0}{y_0}\int y^2 dF = M.$$

The integral $\int y^2 dF$ depends only on the form of the cross section, and is called the **moment of inertia** of the cross section with respect to the neutral axis.

Let the moment of inertia be denoted by I. Then

$$\boldsymbol{I = \int y^2 dF},$$

and, consequently,

(17) $$p_0 = \frac{My_0}{I}.$$

This formula gives the intensity of the normal stress p_0 at the distance y_0 from the neutral axis, due to an external bending moment M. If

p denotes the stress on the extreme fiber and e denotes the distance of this fiber from the neutral axis, then, from (**17**),

$$p = \frac{Me}{I}. \tag{18}$$

Equation (**18**) gives the maximum normal stress on any cross section of a beam, and is the fundamental formula in the common theory of flexure.

Problem 56. Find the moment of inertia of a rectangle of height h and breadth b about a gravity axis * parallel to its base.

Solution.
$$I_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} (bdy)\, y^2 = \frac{by^3}{3} \Big/_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{bh^3}{12}.$$

Problem 57. Find the moment of inertia of a triangle of base b and altitude h about a gravity axis parallel to its base.

Problem 58. Find the moment of inertia of a circle of diameter d about a gravity axis.

Problem 59. The external moment acting on a rectangular section 12 in. deep and 4 in. wide is 30,000 ft. lb. Find the stress on the extreme fiber.

Solution. $M = 30{,}000$ ft. lb. $= 360{,}000$ in. lb.,

$$I_z = \frac{bh^3}{12} = 576 \text{ in.}^4.$$

$$\therefore\ p = \frac{Me}{I} = 3750 \text{ lb./in.}^2.$$

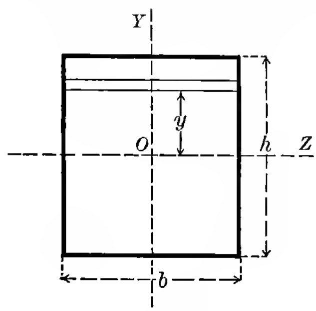


FIG. 23

44. Moment of resistance. The **moment of resistance** is defined as the moment of the internal stresses which balances the external moment M. According to this definition the moment of resistance is simply

$$\frac{pI}{e},$$

since $\frac{pI}{e} = M$. Therefore, if p is the maximum allowable unit stress for any material, the moment of resistance $\frac{pI}{e}$ determines the maximum external bending moment which can be safely carried by a beam of this material.

* In what follows, "gravity axis" will be used as an abbreviation for "axis through the center of gravity."

For instance, consider an oak beam 8 in. deep and 4 in. wide. From Table I, the ultimate compressive strength for timber may be taken as 7000 lb./in.^{2}, and the ultimate tensile strength as 10,000 lb./in.^{2} Therefore, using a factor of safety of 8, the safe unit stress is $p = 875$ lb./in.^{2} For the beam under consideration $I = 170.7$ in.^{4} and $e = 4$ in. Consequently, the maximum bending moment which the beam can be expected to carry safely is 37,340 in. lb., or 3112 ft. lb.

Problem 60. Find the moment of resistance of a circular cast-iron beam 6 in. in diameter.

Problem 61. Find the moment of resistance of a Carnegie steel I-beam, No. B 1, weighing 80 lb./ft.

Problem 62. Compare the moments of resistance of a rectangular beam 8 in. × 14 in. in cross section, when placed on edge and when placed on its side.

45. Section modulus. In Article 43 the moment of inertia was defined as the integral

$$I = \int y^2 dF.$$

From this definition it is apparent that the moment of inertia depends for its value solely on the *form* of the cross section. Since it is independent of all other considerations, it may therefore be called the **shape factor** in the strength of materials.

Since e denotes the distance of the extreme fiber of a beam from the neutral axis, the ratio $\frac{I}{e}$ is also a function of the shape of the cross section, and for this reason is called the **section modulus.** Let the section modulus be denoted by S. Then $S = \frac{I}{e}$, and the expression for the moment of resistance becomes

$$M = pS.$$

This expresses the fact that the strength of a beam depends jointly on the form of cross section and the ultimate strength of the material.

Problem 63. Find the section moduli for the sections given in Problems 56, 57, and 58 respectively.

Problem 64. Compare the section moduli for a rectangle 10 in. high and 4 in. wide, and for one 4 in. high and 10 in. wide.

46. Theorems on the moment of inertia. The following is a summary of the most useful theorems concerning the moment of inertia. The proofs can be found in any standard text-book on mechanics.

(A) Let I_g denote the moment of inertia of any cross section with respect to a gravity axis (see footnote, p. 39), I_n the moment of inertia

of the same section with respect to any parallel axis, c the distance between the two axes, and F the area of the cross section. Then

$$I_n = I_g + Fc^2. \tag{19}$$

(B) Every section has two axes through its center of gravity, called **principal axes**, such that for one of these the moment of inertia is a maximum, and for the other is a minimum. Let the principal axes be taken for the axes of Y and Z respectively. Then if I_y and I_z denote the moments of inertia of the section with respect to these axes, and I_α denotes the moment of inertia with respect to an axis inclined at an angle α to the axis of Z,

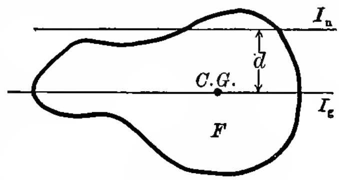


FIG. 24

$$I_\alpha = I_z \cos^2 \alpha + I_y \sin^2 \alpha.* \tag{20}$$

(C) The moment of inertia of a compound section about any axis is equal to the sum of the moments of inertia about this axis of the various parts of which the compound section is composed.

(D) The moment of inertia of any section with respect to an axis through its center of gravity and perpendicular to its plane is called the **polar moment of inertia**. The polar moment of inertia is defined by the equation

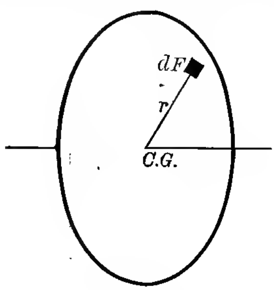


FIG. 25

$$I_p = \int r^2 dF,$$

where r is the distance of the infinitesimal area dF from the center of gravity of the section.

Since $r^2 = y^2 + z^2$,

$$\int r^2 dF = \int y^2 dF + \int z^2 dF, \text{ whence}$$

$$I_p = I_y + I_z. \tag{21}$$

(E) Let I_1 and I_2 denote the moments of inertia of any section with respect to its principal axes. Then $I_p = I_1 + I_2$, and, consequently,

* If the axes of Y and Z are not principal axes, then
$$I_\alpha = I_z \cos^2 \alpha + I_y \sin^2 \alpha - \int\int yz\, dy\, dz.$$

(22) $$I_y + I_z = I_1 + I_2;$$

that is to say, the sum of the moments of inertia with respect to any two rectangular axes in the plane of the section is constant.

(*F*) The numerical value of the moment of inertia is expressed as the fourth power of a unit of length. Therefore the quantity $\frac{I}{F}$ is the square of a length called the **radius of gyration**, and will be denoted by t. The radius of gyration is thus defined by the equation

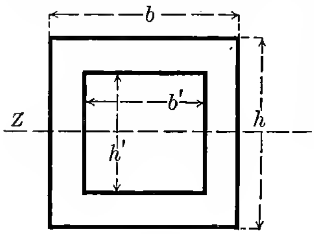


FIG. 26

(23) $$t = \sqrt{\frac{I}{F}};$$

that is to say, the square of the radius of gyration is the mean of the squares of the distances of all the elements of the figure from the axis.

The meaning to be attached to the radius of gyration is that if the total area of the figure was concentrated in a single point at a distance t from the axis, the moment of inertia of this single particle about this axis would be equal to the given moment of inertia.

Problem 65. Find the moment of inertia of the rectangle in Problem 56 about its base, and also the corresponding radius of gyration.

Solution. $I_z = \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}, \quad t_z = \frac{h}{\sqrt{3}}.$

Problem 66. Find the moment of inertia of the above rectangle about a gravity axis inclined at an angle of 30° to its base.

Problem 67. Find the moment of inertia of a rectangular strip, such as that shown in Fig. 26, about a gravity axis parallel to its base.

Problem 68. Prove that the moment of inertia of a T-shape, such as that shown in Fig. 27, about a gravity axis parallel to the base is given by the expression

$$I_z = \frac{b'h^3 + bh'^3 - (b - b')\,d^3}{3}.$$

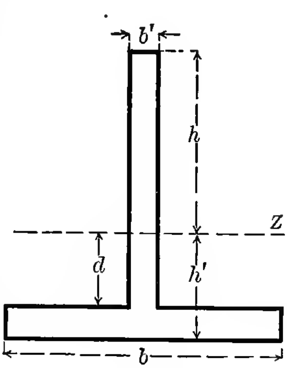


FIG. 27

Problem 69. Find the polar moment of inertia and radius of gyration of a circle of diameter d about an axis through its center.

47. Graphical method of finding the moment of inertia. If the boundary of a given cross section is not composed of simple curves such as straight lines and circles, it is often difficult to find the moment of inertia by means of the calculus. When such difficulties arise the following graphical method may be used to advantage.

To explain the method consider a particular case, such as the rail shape shown in Fig. 28, and suppose that it is required to find the center of gravity of the section, and also its moment of inertia about a gravity axis perpendicular to the web. The first step is to draw two lines, AB and CD, parallel to the required gravity axis, at any convenient distance apart, say l.

If the section is symmetrical about any axis, such as OY in the figure, it is sufficient to consider the portion on either side of this axis, say the part on the right of OY in the present case.

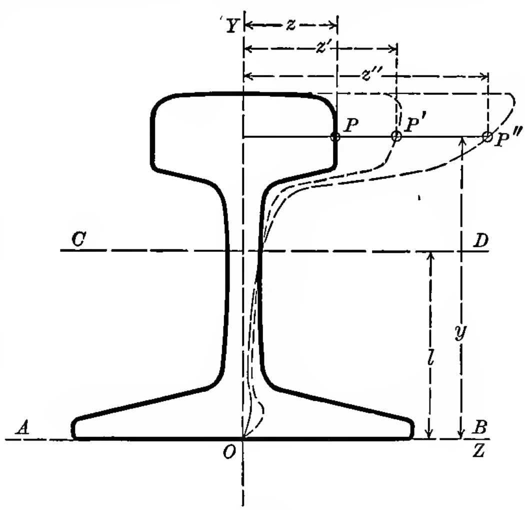


Fig. 28

Now suppose that the cross section is divided into narrow strips parallel to AB and CD; let z denote the length of one of these strips, and dy its width. Then, if for each value of z a length z' is found, such that

$$z' = z\frac{y}{l},$$

any point P on the boundary of the original section, with coördinates z and y, will be transformed into a point P' with coördinates z' and y.

Suppose this process is carried out for a sufficient number of points, and that the points P' so obtained are joined by a curve, as shown by the dotted line in Fig. 28. Let F denote the area of the original curve and F' the area of the transformed curve, both of which can

easily be measured by means of a planimeter. Also let N denote the **static moment** of the original section with respect to the line AB, where the static moment — an area with respect to any axis — is defined by the integral

$$N = \int y dF,$$

in which y is the distance of an infinitesimal area dF from the given axis. The static moment is thus equal to the area of the section multiplied by the distance of its center of gravity from the given axis. Then

$$N = \int y dF = \int yz dy = l \int z' dy = lF'.$$

But, from the above definition,

$$N = cF,$$

where c is the distance of the center of gravity of the original section from the line AB. Therefore $cF = lF'$; whence

$$c = l\frac{F'}{F},$$

which determines the position of the center of gravity.

To find the moment of inertia, make a second transformation by constructing for each z' a value z'', such that

$$z'' = z'\frac{y}{l}.$$

Then the points P' on the first transformed curve are transformed into a series of points P'' on another curve, shown by the broken line in Fig. 28. Let the area of this second curve be denoted by F''. Then, since $z'' = z'\frac{y}{l}$, and $z' = z\frac{y}{l}$, we have $z'' = z\frac{y^2}{l^2}$. Consequently,

$$I = \int y^2 dF = \int y^2 z dy = l^2 \int z'' dy = l^2 F'',$$

which gives the moment of inertia of the original section with respect to the line AB.

If the moment of inertia I_g with respect to a gravity axis is required, then, since by Article 46 (A), $I = I_g + c^2F$, we have $I_g = I - c^2F$; and hence, by substituting the values of I and c from the above,

$$I_g = l^2\left(F'' - \frac{F'^2}{F}\right).$$

The above method is due to Nehrs, and furnishes an easy method of calculating the moment of inertia of any cross section by simply measuring the area F of the original section and the area F', F'' of the transformed sections by means of a planimeter, and then substituting these values in the above formulas.

48. Moment of inertia of non-homogeneous sections. The standard formula for calculating the stress in beams, $p = \frac{Me}{I}$, assumes that the material of which the beam is composed is homogeneous throughout. If, then, a beam is composed of two different materials, such, for instance, as concrete and steel, it is necessary to modify this formula somewhat before applying it.

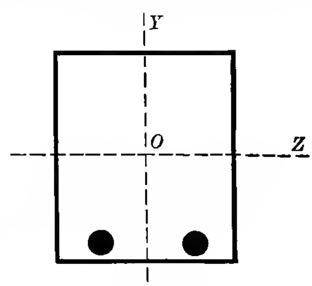


FIG. 29

To exemplify this, consider a rectangular concrete beam, reënforced by steel rods near the bottom, as shown in cross section in Fig. 29. Let p_c and p_s denote the stresses on a fiber of concrete and of steel respectively, at the same distance y from the neutral axis, and let E_c and E_s denote the moduli of elasticity for concrete and steel. Then, by Hooke's law,

$$\frac{p_s}{E_s} = ky = \frac{p_c}{E_c};$$

whence

$$p_s = \frac{E_s}{E_c} p_c.$$

Therefore, if dF is an infinitesimal area of steel at the distance y from the neutral axis, the moment of the stress acting on this area is

$$yp_s dF = y\frac{E_s}{E_c} p_c dF = yp_c\left(\frac{E_s}{E_c} dF\right).$$

Consequently, the intensity of the fiber stress can be considered to vary directly as its distance from the neutral axis over the entire cross section of the beam, provided the area of the steel is increased in the ratio $\frac{E_s}{E_c}$. If, then, the depth is kept constant, the breadth must be increased in this ratio, and the cross section thus obtained

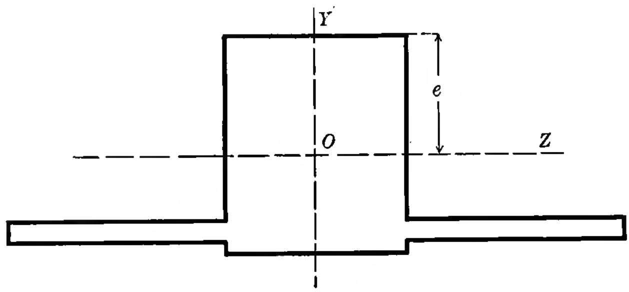


FIG. 30

will appear as shown in Fig. 30. Therefore, if I_c denotes the moment of inertia of this modified section, the stress in the extreme fiber is given by the formula

$$p = \frac{Me}{I_c}.$$

FIG. 31

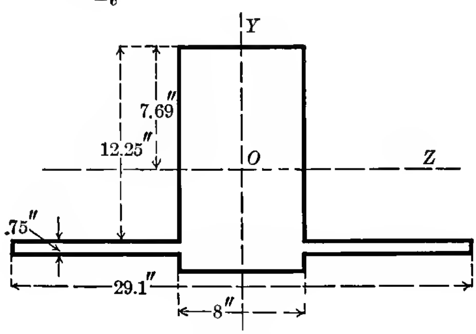


FIG. 32

Problem 70. A rectangular concrete beam 14 in. deep and 8 in. wide is reënforced by two $\frac{3}{4}$-in. square steel rods placed 1 in. from the bottom, as shown in Fig. 31. Assuming that the ratio of the moduli of elasticity of steel and concrete is $E_s : E_c = 15 : 1$, find the moment of inertia of a cross section of the beam about a gravity axis parallel to the base.

Solution. Increasing the area of the steel in the rate 15 : 1, it becomes 16.9 in.^{2}. The area of the concrete included in the same horizontal strip with the steel is

4.9 in.². Consequently, the breadth of the lower flange of the equivalent homogeneous section is

$$\frac{16.9 + 4.9}{.75} = 29.1 \text{ in.}$$

The distance of the center of gravity of this equivalent section below the top is found to be 7.69 in., and its moment of inertia about the gravity axis OZ is 2269 in.⁴ (Fig. 32).

49. Inertia ellipse. Dividing equation (**20**) by F and expressing the result in terms of the radii of gyration by means of equation (**23**),

$$(\mathbf{24}) \qquad t_\alpha^2 = t_z^2 \cos^2\alpha + t_y^2 \sin^2\alpha,$$

where t_y and t_z are the radii of gyration with respect to the axes of Y and Z respectively, and t_α is the radius of gyration with respect to a gravity axis inclined at an angle α to the axis of Z.

Now let l be a length defined by the relation $\frac{t_y t_z}{t_\alpha} = l$. Then $t_y = \frac{lt_\alpha}{t_z}$, $t_z = \frac{lt_\alpha}{t_y}$; and substituting these values of t_y and t_z in equation (**24**), it becomes

$$t_\alpha^2 = \frac{l^2 t_\alpha^2}{t_y^2} \cos^2\alpha + \frac{l^2 t_\alpha^2}{t_z^2} \sin^2\alpha,$$

or, dividing by t_α^2,

$$1 = \frac{(l \cos\alpha)^2}{t_y^2} + \frac{(l \sin\alpha)^2}{t_z^2}.$$

This is the equation of an ellipse with semi-axes t_y and t_z, called the **inertia ellipse**, the coördinates of any point of the curve being $l \cos\alpha$ and $l \sin\alpha$.

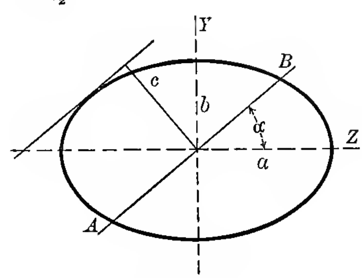


FIG. 33

By means of the inertia ellipse the moment of inertia with respect to any gravity axis AB (Fig. 33) can be obtained as follows.

The equation of a tangent to the ellipse $\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(z'y')$ is $\frac{zz'}{a^2} + \frac{yy'}{b^2} = 1$, or

$$(\mathbf{25}) \qquad zz'b^2 + yy'a^2 - a^2b^2 = 0.$$

It is proved in analytical geometry that in order to reduce the linear equation $Az + By + C = 0$ to the normal form $z \cos \beta + y \sin \beta - c = 0$, it is necessary to divide throughout by $\sqrt{A^2 + B^2}$. Applying this theorem to equation (**25**), it becomes

$$\frac{z'b^2}{\sqrt{z'^2b^4 + y'^2a^4}} z + \frac{y'a^2}{\sqrt{z'^2b^4 + y'^2a^4}} y - \frac{a^2b^2}{\sqrt{z'^2b^4 + y'^2a^4}} = 0,$$

where

$$\frac{z'b^2}{\sqrt{z'^2b^4 + y'^2a^4}} = \cos \beta, \quad \frac{y'a^2}{\sqrt{z'^2b^4 + y'^2a^4}} = \sin \beta, \quad \frac{a^2b^2}{\sqrt{z'^2b^4 + y'^2a^4}} = c.$$

Substituting these values in the expression $a^2 \cos^2 \beta + b^2 \sin^2 \beta$, it becomes

$$a^2 \cos^2 \beta + b^2 \sin^2 \beta = \frac{a^2b^4z'^2}{z'^2b^4 + y'^2a^4} + \frac{b^2a^4y'^2}{z'^2b^4 + y'^2a^4}$$

$$= \frac{a^2b^2(b^2z'^2 + a^2y'^2)}{z'^2b^4 + y'^2a^4} = \frac{a^4b^4}{z'^2b^4 + y'^2a^4} = c^2;$$

whence, since $\beta = \alpha \pm \frac{\pi}{2}$,

$$c^2 = a^2 \cos^2 \beta + b^2 \sin^2 \beta = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha.$$

Since the semi-axes of the inertia ellipse are $a = t_y$ and $b = t_z$, this expression becomes

$$c^2 = t_y^2 \sin^2 \alpha + t_z^2 \cos^2 \alpha,$$

or, comparing this expression with equation (**24**),

$$c = t_\alpha.$$

The radius of gyration corresponding to any gravity axis AB can therefore be found by drawing a tangent to the inertia ellipse parallel to AB, and measuring the distance of this tangent from the center.

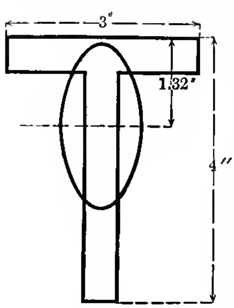


FIG. 34

Since the inertia ellipse is constructed on the principal radii of gyration as semi-axes, it can be drawn on all the ordinary forms of cross section, and when this is done the method given above greatly simplifies the calculation of the moment of inertia with respect to any gravity axis which is not a principal axis.

Problem 71. From the Carnegie handbook of structural steel the principal radii of gyration of T-shape, No. 72, size 3 in. by 4 in., are 1.23 in. and .59 in. Construct the inertia ellipse (Fig. 34).

Problem 72. For a Carnegie I-beam, No. B 7, 15 in. deep and weighing 42 lb./ft., the principal radii of gyration are 5.95 in. for an axis perpendicular to web at center, and 1.08 in. for an axis coincident with web at center. Construct the inertia ellipse.

Problem 73. For a Cambria channel, No. C 21, depth of web 7 in., width of flanges 2.51 in., thickness of web .63 in., the radius of gyration about an axis perpendicular to the web at center is 2.39 in.; the distance of the center of gravity from outside of web is .58 in., and the radius of gyration about an axis through the center of gravity parallel with center line of web is .56 in. Construct the inertia ellipse.

Problem 74. In Problems 68, 69, and 70 determine graphically the radii of gyration about an axis through the center of gravity and inclined at 30° to the major axis of the inertia ellipse.

50. Vertical reactions and shear. Under the assumptions of the common theory of flexure, the external forces acting on a beam all lie in the same vertical plane. Therefore, since the beam is assumed to be in equilibrium, the sum of the reactions of the supports must equal the total load on the beam.

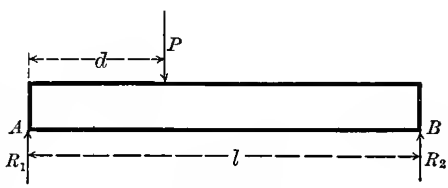


FIG. 35

For instance, consider a simple beam AB of length l, which is supported at the ends and bears a single concentrated load P at a distance d from A (Fig. 35). Let R_1 and R_2 denote the reactions at A and B respectively. Then, from the above,

$$R_1 + R_2 = P.$$

To find the values of R_1 and R_2, take moments about either end, say A. Then

$$R_2 l = Pd;$$

whence

$$R_2 = \frac{Pd}{l}.$$

Also, since

$$R_1 + R_2 = P,$$

$$R_1 = \frac{P(l-d)}{l}.$$

If any cross section of a beam is taken, the stresses acting on this section must reduce to a single force and a moment, as explained in

Article 37. For a simple beam placed horizontally and supporting a system of vertical loads, the plane of the moment is perpendicular to the plane of the section, and the single force is a vertical shear lying in the plane of the section. Therefore, since the portion of the beam on either side of the section must be in equilibrium, the vertical shear is equal to the algebraic sum of the external forces on either side of the section. Thus, if the portion of the beam on the left of the section is considered, the vertical shear on the section is equal to the reaction of the left support minus the sum of the loads on the left of the section.

Problem 75. A beam 10 ft. long bears a uniform load of 300 lb./ft. Find the vertical shear on a section 4 ft. from the left support.

Solution. The total load on the beam is 3000 lb. Therefore, since the load is uniform, each reaction is equal to 1500 lb. The load on the left of the section is $300 \times 4 = 1200$ lb. Therefore the vertical shear on the section is $1500 - 1200 = 300$ lb.

Problem 76. Find the vertical shear at the center and ends of the beam in the preceding problem.

Problem 77. A beam 12 ft. long bears loads of 1, $\frac{1}{2}$, and 3 tons at distances of 2, 5, and 7 ft. respectively from the left support. Find the vertical shear at either end of the beam, and also at a point between each pair of loads.

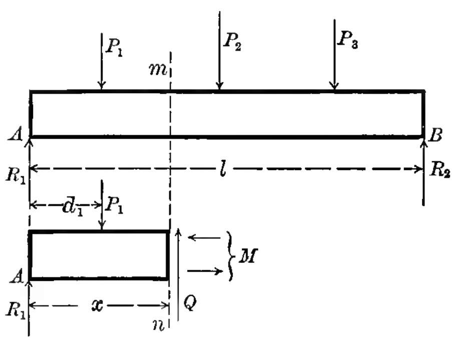


FIG. 36

51. Maximum bending moment. The external bending moment at any point of a beam is defined as the sum of the moments, about the neutral axis of a cross section through the point, of all the external forces on either side of the section. Thus, if the portion of the beam on the left of the section is considered, the external moment at this point is the moment of the reaction of the left support about the neutral axis of the section, minus the sum of the moments of the loads between the left support and the section, about the same neutral axis.

For example, in Fig. 36 the moment of R_1 about the neutral axis of the section mn is R_1x, and the moment of P_1 about the same axis is $P_1(x - d_1)$. Therefore the total external moment acting on the section mn is

$$M = R_1x - P_1(x - d_1).$$

As another example, consider a beam of length l bearing a uniform load of amount w per unit of length. Then the total load on the beam is wl, and each reaction is $\frac{wl}{2}$. Therefore the moment at a point distant x from the left support is

$$M = \frac{wl}{2} \cdot x - wx \cdot \frac{x}{2} = \frac{wx}{2}(l - x).$$

From this relation it is evident that M is zero for $x = 0$ or l, and attains its maximum value for $x = \frac{l}{2}$; that is to say, the bending moment is zero at either end of the beam and a maximum at the center.

From the formula $M = pS$, given in Article 45, it is evident that the maximum value of the stress p occurs where the bending moment M is a maximum. Ordinarily the maximum bending moment produces a greater strain than the maximum shear; therefore the section at which the maximum moment occurs is called the **dangerous section**, since it is the section at which the material is most severely strained, and consequently the one at which rupture may be expected to occur.

In order to find the maximum bending stress in a beam, the formula $M = pS$ is written

$$p = \frac{M}{S}.$$

The maximum bending stress is then obtained at once by simply dividing the maximum bending moment by the section modulus.

Problem 78. A rectangular wooden beam 14 ft. long, 4 in. wide, and 9 in. deep bears a uniform load of 75 lb./ft. Find the position and amount of the maximum bending moment.

Problem 79. Find the maximum bending stress in the beam in the preceding problem.

Problem 80. A Cambria I-beam, No. B 33, which weighs 40 lb./ft., is 15 ft. long and bears a single concentrated load of 5 tons at its center. Find the maximum bending stress in the beam, taking into account the weight of the beam.

52. Bending moment and shear diagrams. In general, the bending moment and shear vary from point to point along a beam. This variation is shown graphically in the following diagrams for several different systems of loading.

(*A*) *Simple beam bearing a single concentrated load P at its center* (Fig. 37). From symmetry the reactions R_1 and R_2 are each equal to $\frac{P}{2}$. Let mn be any section of the beam at a distance x from the left support, and consider the portion of the beam on the left of this

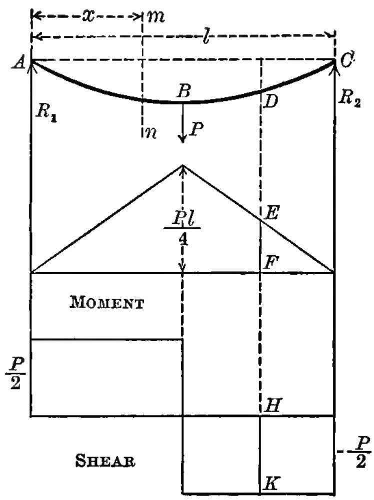


Fig. 37

section. Then the moment at mn is $R_1x\left(=\frac{P}{2}x\right)$ and the shear is $R_1\left(=\frac{P}{2}\right)$. For a section on the right of the center the bending moment is $R_2(l-x)$ and the shear is R_2. Consequently, the bending moment varies as the ordinates of a triangle, being zero at either support, and attaining a maximum value of $\frac{Pl}{4}$ at the center, while the shear is constant from A to B, and also constant, but of opposite sign, from B to C.

The diagrams in Fig. 37 represent these variations in bending moment and shear along the beam under the assumed loading. Consequently, if the ordinates vertically beneath B are laid off to scale to represent the bending moment and shear at this point, the bending moment and shear at any other point D of the beam are found at once from the diagram by drawing the ordinates EF and HK vertically beneath D.

(*B*) *Beam bearing a single concentrated load P at a distance c from one support.*

The reactions in this case are

$$R_1 = \frac{P(l-c)}{l}$$

and

$$R_2 = \frac{Pc}{l}.$$

Hence the bending moment

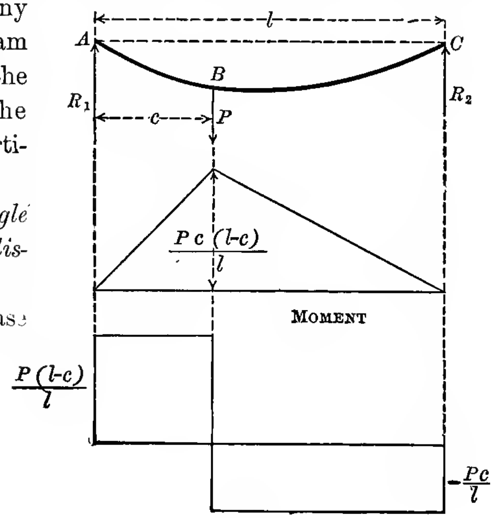


Fig. 38

at a distance x from the left support is

$$R_1 x = \frac{P(l-c)x}{l}$$

provided $x < c$, and

$$R_2(l-x) = \frac{Pc(l-x)}{l}$$

if $x > c$. If $x = c$, each of these moments becomes

$$\frac{Pc(l-c)}{l},$$

and consequently the bending moment and shear diagrams are as shown in Fig. 38.

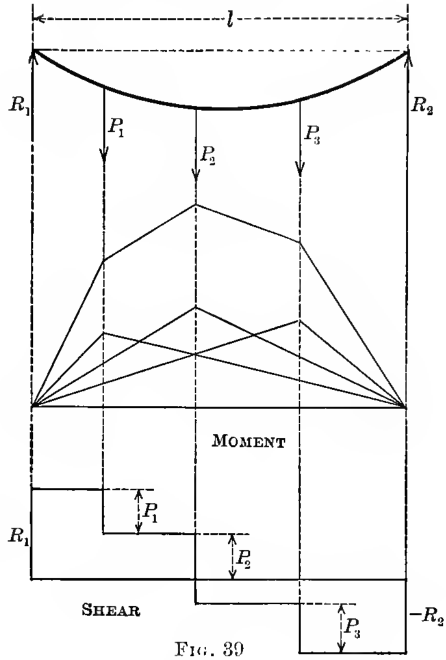


FIG. 39

(*C*) *Beam bearing several separate loads.*

In this case the bending moment diagram is obtained by constructing the diagrams for each load separately and then adding their ordinates, as indicated in Fig. 39.

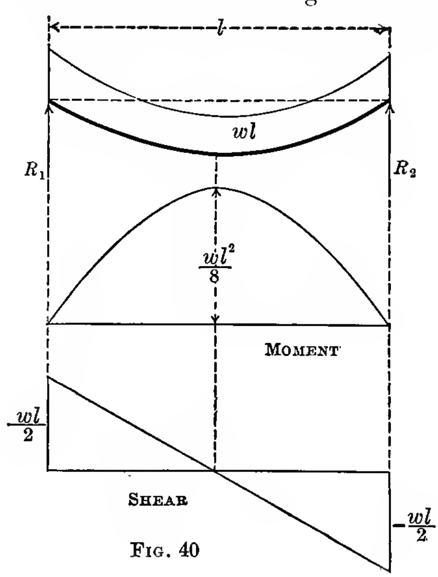


FIG. 40

(*D*) *Beam bearing a continuous uniform load.*

Let the load per unit of length be denoted by w. Then the total load on the beam is wl, and the reactions are

$$R_1 = R_2 = \frac{wl}{2}.$$

Hence at a distance x from the left support the bending moment M_x is

$$M_x = \frac{wl}{2}x - wx\cdot\frac{x}{2} = \frac{w}{2}(lx - x^2).$$

The bending moment diagram is therefore a parabola. For $x = \frac{l}{2}$, $M_x = \frac{wl^2}{8}$, which is its maximum value. The bending moment and shear diagrams are therefore as represented in Fig. 40.

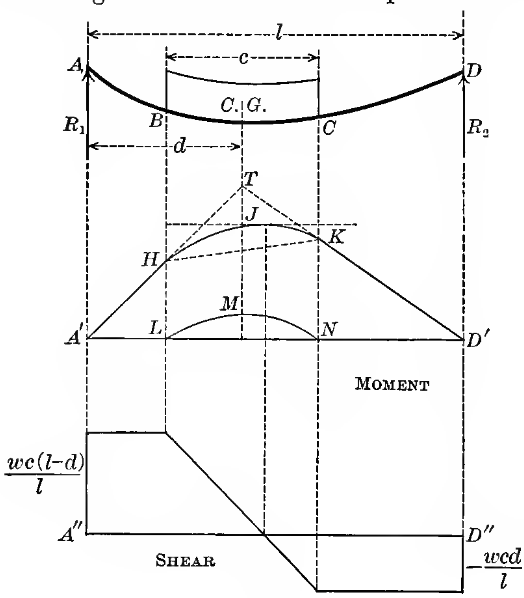


FIG. 41

(*E*) *Beam bearing uniform load over part of the span.*

Let the load extend over a distance c and be of amount w per unit of length. Then the total load is wc. The reactions of the supports are the same as though the load was concentrated at its center of gravity G. Therefore, if d denotes the distance of G from the left support,

$$R_2 = \frac{wcd}{l} \quad \text{and} \quad R_1 = \frac{wc(l-d)}{l}.$$

Also, the bending moment diagrams for the portions AB and CD are the same as though the load was concentrated at G, and are therefore the straight lines $A'H$ and $D'K$, intersecting in the point T vertically beneath G (Fig. 41).

From B to C there is an additional bending moment due to the uniform load on this portion of the beam. Thus, if LMN is the parabolic moment diagram for a beam of length LN or c, the ordinates to the line HK must be increased by those to the parabola LMN, giving as a complete moment diagram the line $A'HJKD'$.

Analytically, if x denotes the distance of any section from the left support, the equations of the three portions $A'H$, HJK, and KD' of the moment diagram are

$$M_{AB} = R_1 x = \frac{wc(l-d)x}{l}, \quad \text{for} \quad 0 \leqq x \leqq d - \frac{c}{2};$$

$$M_{BC} = R_1 x - \frac{w\left(x - d + \dfrac{c}{2}\right)^2}{2} = \frac{wc(l-d)x}{l} - \frac{w\left(x - d + \dfrac{c}{2}\right)^2}{2},$$

$$\text{for} \quad d - \frac{c}{2} \leqq x \leqq d + \frac{c}{2};$$

$$M_{CD} = R_2 x = \frac{wcd(l-x)}{l}, \quad \text{for} \quad d + \frac{c}{2} \leqq x \leqq l.$$

Problem 81. Construct the bending moment and shear diagrams for a cantilever* bearing a single concentrated load P at the end.

Problem 82. Construct the bending moment and shear diagrams for a simple beam bearing two equal concentrated loads at equal distances from the center.

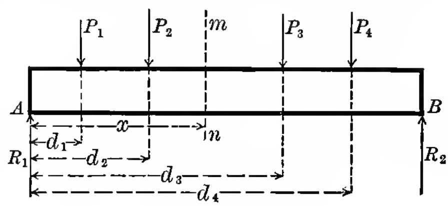


FIG. 42

53. Relation between shear and bending moment. Consider a beam bearing several concentrated loads P_1, P_2, etc., at distances d_1, d_2, etc., from the left support. Take any section mn at a distance x from the left support, and consider the portion of the beam on the left of this section. Then if Q denotes the total shear on this section,

$$Q = R_1 - \sum_0^x P.$$

Also, the bending moment at mn is

$$M = R_1 x - \sum_0^x P(x - d),$$

where the summations include all the loads between A and the section mn.

* A cantilever is a beam which is framed into a wall or other support at one end and projects outward from this support.

Differentiating M with respect to x,

$$\frac{dM}{dx} = R_1 - \sum_0^x P.$$

Therefore

(26) $$\frac{dM}{dx} = Q;$$

that is to say, *the shear at any point of a beam is the first differential coefficient of the bending moment at that point.*

If the beam is uniformly loaded, as in (D) of the preceding article, $Q = R_1 - wx$ and $M = R_1x - \frac{wx^2}{2}$, from which equation (26) results as before.

From equation (26) it follows that if the bending moment is constant the shear is zero; and conversely, if the shear is zero the bending moment is constant. But $\frac{dM}{dx} = 0$ is the condition that the bending moment shall be either a maximum or a minimum. Consequently, *at a point where the bending moment passes through a maximum or minimum value the shear is zero; and conversely.* This theorem is illustrated by the bending moment and shear diagrams in the preceding paragraph.

54. Designing of beams. In designing beams the problem is to find the transverse dimensions of a beam of given length and given material, so that it shall bear a given load with safety.

In order to solve this problem, the formula $M = pS$ is written

$$\frac{M}{p} = S.$$

Then, from the given loading, the maximum value of M is determined, and by dividing the ultimate strength of the material by the proper factor of safety the safe unit stress p becomes known. The quotient of these two gives the section modulus of the required section.

In the handbooks issued by the various structural iron and steel companies, the section moduli of all the standard sections are tabulated. If, then, the beam is to be of a standard shape, its size is found by simply looking in the tables for the value of S which corresponds most closely to the calculated value $\frac{M}{p}$, the value chosen

being equal to or greater than the calculated value in order to insure safety.

If the section of the beam is to be of a shape not listed in the handbooks, the dimensions of the section must be found by trial. Thus a section of the required shape is assumed, and its section modulus calculated from the relation

$$S = \frac{I}{e}.$$

If the value of S thus found is too great or too small, the dimensions of the section are decreased or increased, and S again calculated. Proceeding in this way, the dimensions of the section are changed until a value of S is found which is approximately equal to the calculated value $\frac{M}{p}$.

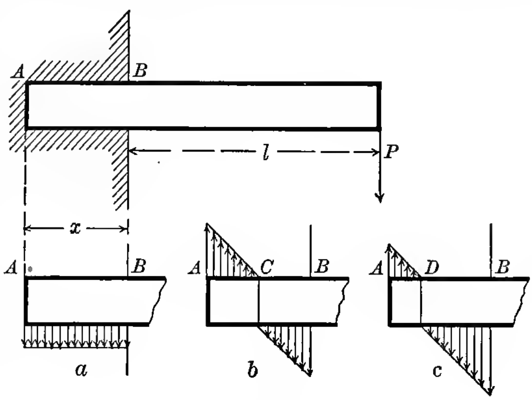


FIG. 43

Problem 83. Design a steel I-beam, 10 ft. long, to bear a uniform load of 1500 lb./ft., neglecting its own weight.

Problem 84. A built beam is to be composed of two steel channels placed on edge and connected by latticing. What must be the size of the channels if the beam is to be 18 ft. long and bear a load of 10 tons at its center, the factor of safety being given as 4?

Problem 85. Compare the strength of a pile of 10 boards, each 14 ft. long, 1 ft. wide, and 1 in. thick, when the boards are piled horizontally, and when they are placed close together on edge.

Problem 86. Design a rectangular wooden cantilever to project 4 ft. from a wall and bear a load of 500 lb. at its end, the factor of safety being 8.

Problem 87. A rectangular cantilever projects a distance l from a brick wall and bears a single concentrated load P at its end. How far must the inner end of the cantilever be imbedded in the wall in order that the pressure between this end and the wall shall not exceed the crushing strength of the brick?

Solution. Let b denote the width of the beam and x the distance it extends into the wall. For equilibrium the reaction between the beam and the wall must consist of a vertical force and a moment. If p_a denotes the intensity of the vertical

stress, and it is assumed to be uniformly distributed over the area bx, $p_a bx = P$; whence $p_a = \frac{P}{bx}$ (see Fig. 43, a).

Similarly, let p_b denote the maximum intensity of the stress forming the stress couple. Then, taking moments about the center C of the portion AB, since the stress forming the couple is also distributed over the area bx, we have

$$I = \frac{bx^3}{12}, \qquad e = \frac{x}{2}, \qquad \text{and} \qquad M = P\left(l + \frac{x}{2}\right).$$

Therefore, substituting in the formula $p = \frac{Me}{I}$, we have

$$p_b = \frac{P\left(l + \frac{x}{2}\right)\frac{x}{2}}{\frac{bx^3}{12}} = \frac{6\,P\left(l + \frac{x}{2}\right)}{bx^2}.$$

Consequently,

$$p_{\substack{\max\\ \min}} = p_b \pm p_a = \frac{6\,P\left(l + \frac{x}{2}\right)}{bx^2} \pm \frac{P}{bx};$$

whence

$$p_{\max} = \frac{2\,P}{bx}\left(2 + \frac{3\,l}{x}\right),$$

and

$$p_{\min} = \frac{2\,P}{bx}\left(1 + \frac{3\,l}{x}\right).^{*}$$

As a numerical example of the above, let $l = 5$ ft. $\doteq 60$ in., $P = 200$ lb., $b = 4$ in., and $p = 600$ lb./in.^{2} (for ordinary brick work). Solving the above equation by the formula for quadratics,

$$x = \frac{2\,P \pm \sqrt{4\,P^2 + 6\,bpPl}}{bp};$$

whence, by substituting the above values,

$$x = 5.6 \text{ in.}$$

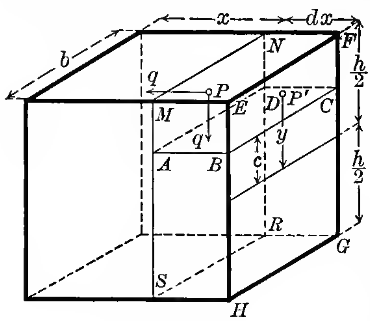


Fig. 44

†55. Distribution of shear over rectangular cross section. Consider a cross section of a rectangular beam at a distance x from the left support, as $MNRS$ in Fig. 44, and let P be a point in this cross section at a distance y from the neutral axis. Then, by equation (**17**), Article 43, the unit normal stress at P is $p = \frac{My}{I}$. If the cross

* Bach, *Elasticität u. Festigkeitslehre*, p. 430.

† For a brief course in the Strength of Materials the remainder of this chapter may be omitted.

section is moved from this position parallel to itself a distance dx, say to the position $EFGH$ in the figure, the rate of change of p with respect to x is

$$\frac{dp}{dx} = \frac{dM}{dx} \cdot \frac{y}{I} = \frac{y}{I} Q. \tag{27}$$

The difference between the normal stresses acting on these two adjacent cross sections tends to shove the point P in a direction parallel to the axis of the beam, and this tendency is resisted by a shearing stress of intensity q at P, also parallel to the axis of the beam. Therefore, since the resultant normal stress on the area $BCEF$ is $\int_c^{\frac{h}{2}} dp \cdot dF$, and the resultant shearing stress on the area $ABCD$ is $qbdx$,

$$\int_c^{\frac{h}{2}} dp \cdot dF = qbdx.$$

Substituting the value of dp from equation (**27**),

$$\frac{Qdx}{I} \int_c^{\frac{h}{2}} ydF = qbdx;$$

whence

$$q = \frac{Q}{bI} \int_c^{\frac{h}{2}} ydF. \tag{28}$$

Formula (**28**) applies to any cross section bounded by parallel sides.

In Article 23 it was proved that whenever a shearing stress acts along any plane in an elastic solid, there is always another shearing stress of equal intensity acting at the same point in a plane at right angles to the first. Consequently, formula (**28**) also gives the intensity of the stress at any point P in a direction perpendicular to the neutral axis of the section.

For a *rectangular cross section*

$$\int_c^{\frac{h}{2}} ydF = \int_c^{\frac{h}{2}} bydy = b\left(\frac{h^2}{8} - \frac{c^2}{2}\right),$$

and hence

$$\boldsymbol{q = \frac{Q}{I}\left(\frac{h^2}{8} - \frac{c^2}{2}\right).} \tag{29}$$

From equation (**29**), it is evident that for rectangular sections the shear is zero at the top and bottom of the beam $\left(\text{where } c = \frac{h}{2}\right)$ and increases toward the center as the ordinates to a parabola. For $c = 0$, q attains its maximum value, namely, $q = \frac{Qh^2}{8\,I}$ (Fig. 45). At the top and bottom where the normal bending stress is greatest the shear is zero, and at the center where the normal stress is zero the shear is a maximum.

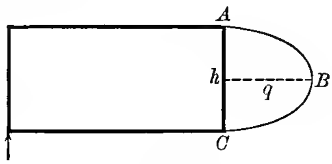


FIG. 45

Since the area of the parabola ABC is $\frac{2}{3}hq$, the average stress is $\frac{2}{3}hq/h = \frac{2}{3}q$, and consequently the maximum unit stress q is $\frac{3}{2}$ average unit stress.

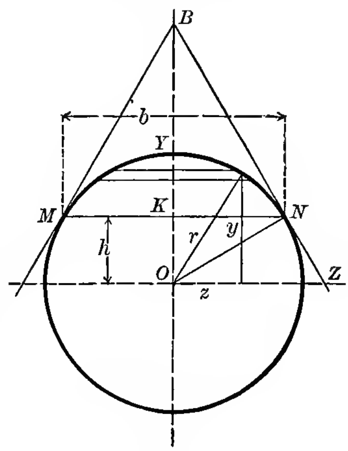


FIG. 46

56. Distribution of shear over circular cross section. For a rectangular cross section the shear parallel to the neutral axis is zero, but for a circular cross section this is not the case. Let Fig. 46 represent a circular cross section, say the cross section of a rivet subjected to a vertical shear, and let it be required to find the direction and intensity of the shear at the extremity N of a horizontal line MN. If the stress at N has a normal component, that is, a component in the direction ON, it must have a component of equal amount through N perpendicular to the plane of the cross section, that is, in the direction of the axis of the rivet (Article 23). Consequently, since the rivet receives no stress in the direction of its axis, the stress at N can have no normal component and is therefore tangential.

Similarly, the stress at M is tangential, and since the line MN is horizontal, the tangents at M and N must meet at some point B on the vertical diameter, which is taken for the Y-axis. The stress at any point K on the Y-axis must act in the direction of this axis, and

therefore also pass through B. For any other point of MN it is approximately correct to assume that the direction of the stress also passes through B.

Therefore, in order to determine the direction and intensity of the shear at any point of a circular cross section, a chord is drawn through the point perpendicular to the direction of the shear and tangents drawn at its extremities, thus determining a point such as B in Fig. 46. Assuming the axes as in Fig. 46, the vertical shear acting at the point is then calculated by formula (28), where, in the present case, b is the length of the chord and the integral is extended over the segment above the chord. The horizontal component of the shear is then determined by the condition that the resultant of these two components must pass through B.

The amount of the component and resultant shears acting at any point can be calculated as follows.

For a strip parallel to the Z-axis, $dF = zdy$, and $z = \sqrt{r^2 - y^2}$. Therefore

$$\int_h^r ydF = 2\int_h^r \sqrt{r^2 - y^2}\, ydy = -\frac{2}{3}(r^2 - y^2)^{\frac{3}{2}} \Big/_h^r = \frac{b^3}{12}.$$

The vertical component of the shear is, therefore,

$$q_y = \frac{Q}{bI}\left(\frac{b^3}{12}\right) = \frac{Qb^2}{3\,\pi r^4}.$$

Let KB and KN, Fig. 46, represent in magnitude and direction the vertical and horizontal components of the shear acting at N. Then, from the similar triangles KNB and KNO,

$$\frac{KN}{KB} = \frac{KO}{KN}, \quad \text{or} \quad \frac{q_z}{q_y} = \frac{h}{\frac{b}{2}};$$

whence

$$q_z = \frac{2\,q_y h}{b} = \frac{2\,Qbh}{3\,\pi r^4}.$$

Since $\overline{BN}^2 = \overline{BK}^2 + \overline{KN}^2$, the resultant shear at N is

$$q = \sqrt{q_y^2 + q_z^2} = \frac{Qb}{3\,\pi r^4}\sqrt{b^2 + 4\,h^2},$$

or, since $\frac{b^2}{4} + h^2 = r^2$,

$$q = \frac{2\,Qb}{3\,\pi r^3}.$$

In this equation q is proportional to b, and hence the maximum value of q is at the center where $b = 2\,r$. Hence

$$q_{\max} = \frac{4\,Q}{3\,\pi r^2}.$$

The maximum unit shear on a circular cross section is therefore equal to $\frac{4}{3}$ of its average value.

57. Cases in which shear is of especial importance. In Article 53 it was shown that at points where the normal bending stress is a maximum the shear is zero. For this reason it is usually sufficient to dimension a beam so as to carry the maximum bending stress safely without regard to the shear. However, in certain cases, of which the following are examples, it is necessary to calculate the shear also, and combine it with the bending stress.

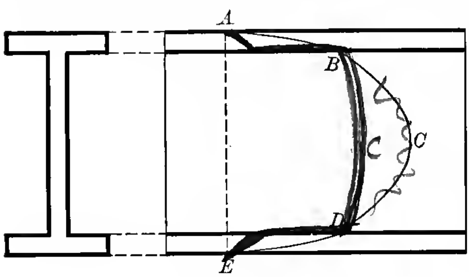


FIG. 47

For an I-beam the static moment $\int y dF$ is nearly as great directly under the flange as for a section through the neutral axis; and therefore, by formula (**28**), the shear is very large at this point, as shown on the shear diagram in Fig. 47. Hence the shear and bending stress are both large under the flange, and the resultant stress at this point may, in some cases, exceed that at the outer fiber.

Again, if a beam is very short in comparison with its depth, or if the material of which it is made offers small resistance to shear in certain directions, as in the case of a wooden beam parallel to the grain, a special investigation of the shear must be made. For instance, consider a rectangular wooden beam of length l, breadth b, and depth h, bearing a single concentrated load P at its center. Then the total

shear on any section is $\frac{P}{2}$, and the maximum bending moment is $\frac{Pl}{4}$. Hence the maximum unit normal stress is

$$p = \frac{M}{I}\cdot\frac{h}{2} = \frac{3\,Pl}{2\,bh^2}.$$

Also, since $Q = \frac{P}{2}$ and $\int_0^{\frac{h}{2}} ydF = \frac{bh^2}{8}$, the maximum unit shear is

$$q = \frac{Q}{bI}\int ydF = \frac{3\,P}{4\,bh}.$$

Now let κ denote the ratio between the tensile strength in the direction of the fiber and the shearing strength parallel to the fiber. Then, in order that the beam shall be equally safe against normal and shearing stress, $p = \kappa q$, or

$$\frac{3\,Pl}{2\,bh^2} = \kappa\,\frac{3\,P}{4\,bh};$$

whence

$$\frac{2\,l}{h} = \kappa.$$

In general, κ is not greater than 10. If $\kappa = 10$, $l = 5\,h$. Consequently, if the length of a beam is greater than 5 times its depth, the shear is not likely to cause rupture.

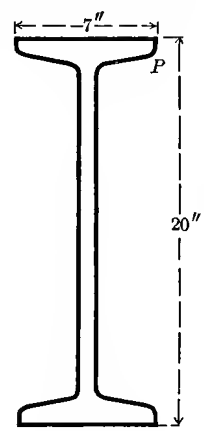


Fig. 48

Problem 88. The bending moment and shear at a certain point in a Carnegie I-beam, No. B 2, of the dimensions given in Fig. 48, are $M = 200{,}000$ ft. lb. and $Q = 15{,}000$ lb. respectively. Calculate the maximum normal stress and the equivalent stress for a point directly under the flange, and compare these values with the normal stress in the extreme fiber.

Solution. From the Carnegie handbook, the moment of inertia of this section about a neutral axis perpendicular to the web is $I = 1466.5$ in.^{4}. Consequently, the normal stress in the extreme fiber is

$$p_{\max} = \frac{Me}{I} = \frac{2{,}400{,}000\,(10)}{1466.5} = 16{,}365 \text{ lb./in.}^2,$$

and the normal stress at a point P under the flange is

$$p = \frac{2{,}400{,}000\,(9.35)}{1466.5} = 15{,}300 \text{ lb./in.}^2.$$

Neglecting the rounded corners,

$$\int_k^{\frac{h}{2}} ydF = \int_{9.35}^{10} 7\, ydy = 44 \text{ in.}^3.$$

Consequently, from formula (**28**), the unit shear at P is

$$q = \frac{Q}{bI}\int_k^{\frac{h}{2}} ydF = \frac{15{,}000\,(44)}{7\,(1466.5)} = 64 \text{ lb./in.}^2.$$

At the point P, therefore, $p_x = 15{,}300$ lb./in.^{2}, $p_y = 0$, and $q = 64$ lb./in.^{2}. Hence, from formula (**7**), Article 26,

$$p_{\max} = \frac{p_x}{2} + \frac{1}{2}\sqrt{4\,q^2 + p_x^2} = 15{,}304 \text{ lb./in.}^2.$$

To calculate the equivalent stress it is necessary to find the principal stresses, which are, from the above,

$$p_1 = 15{,}304 \text{ lb./in.}^2 \quad \text{and} \quad p_2 = -2 \text{ lb./in.}^2.$$

Hence, from formula (**14**), Article 35, for $m = 3\frac{1}{3}$ the equivalent stress at P is

$$p_e = 15{,}305 \text{ lb./in.}^2.$$

58. Oblique loading. If, for any cross section, the plane of the external bending moment does not pass through a principal axis of the section, the loading is said to be *oblique*. In this case the bending moment M can be resolved into components parallel to the principal axes, namely, $M \cos \alpha$ and $M \sin \alpha$, where α is the angle which the plane containing M makes with one of the principal axes.

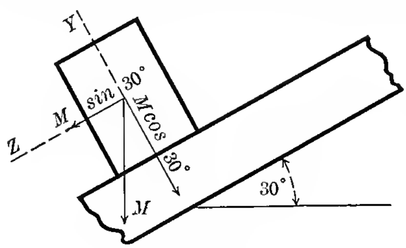


FIG. 49

For materials which conform to Hooke's law it has been found that the stress due to several sets of external forces can be calculated for each set separately and then combined into a single resultant. This is called the **law of superposition**. Applying this law to the present case,

$$(30) \qquad p = \frac{M\cos\alpha}{I_z}\cdot e_z + \frac{M\sin\alpha}{I_y}\cdot e_y = \frac{M\cos\alpha}{S_z} + \frac{M\sin\alpha}{S_y},$$

where e_y, e_z are the distances of the extreme fibers of the beam from the axes of Y and Z respectively, and S_y, S_z are the corresponding section moduli.

Problem 89. In an inclined railway the angle of inclination with the horizontal is 30°. The stringers are 10 ft. 6 in. apart, inside measurement, and the rails are placed 1 ft. inside the stringers. The ties are 8 in. deep and 6 in. wide, and the maximum load transmitted by each rail to one tie is 10 tons. Calculate the maximum normal stress in the tie.

Solution. The bending moment is the same for all points of the tie between the rails, and is 20,000 ft. lb. From Problem 66, $S_z = 64$ in.³ and $S_y = 48$ in.³. Therefore, from equation (**30**),

$$p_{\max} = \frac{240{,}000\left(\frac{\sqrt{3}}{2}\right)}{64} + \frac{240{,}000\left(\frac{1}{2}\right)}{48} = 5744 \text{ lb./in.}^2.$$

59. Eccentric loading. If the external forces acting on any cross section reduce to a single force P, perpendicular to the plane of the section, but not passing through its center of gravity, this force is called an **eccentric load.** Let B denote the point of application of the eccentric load P, and let $y'z'$ denote the coördinates of B. Then the eccentric force P acting at B can be replaced by an equal and parallel force acting at the center of gravity C of the section, and a moment whose plane is perpendicular to the section. This moment can then be resolved into two components parallel to the principal axes, of amounts Py' and Pz' respectively. Therefore, by the law of superposition, the intensity of the stress at any point (y, z) of the cross section is

$$p = \frac{P}{F} + \frac{Pz'}{I_y}\cdot z + \frac{Py'}{I_z}y;$$

or, since $I = Ft^2$,

$$p = \frac{P}{F}\left(1 + \frac{zz'}{t_y^2} + \frac{yy'}{t_z^2}\right).$$

At the neutral axis the stress is zero, and consequently $1 + \frac{zz'}{t_y^2} + \frac{yy'}{t_z^2}$ must be zero; or, since the semi-axes of the inertia ellipse are $a = t_y$ and $b = t_z$, this condition becomes

$$\text{(31)} \qquad \frac{zz'}{a^2} + \frac{yy'}{b^2} = -1.$$

This condition must be satisfied by every point on the neutral axis, and is therefore the **equation of the neutral axis.** To each pair of values of y' and z', that is, to each position of the point of application B of the eccentric load, there corresponds one and only one position of the neutral axis.

If the point B lies on the ellipse $\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1$, its coördinates must satisfy this equation, and, consequently,

$$\frac{z'^2}{a^2} + \frac{y'^2}{b^2} = 1. \tag{32}$$

In this case the neutral axis passes through a point on the ellipse diametrically opposite to B; for if $-z'$, $-y'$ are substituted for y and z in equation (**31**), it is evident that the condition (**32**) is satisfied.

The tangent to the ellipse $\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $-z'$, $-y'$ is $\frac{zz'}{a^2} + \frac{yy'}{b^2} = -1$, which is identical with equation (**31**). Consequently, if B lies on the inertia ellipse, the neutral axis corresponding to B is tangent to the ellipse at the point diametrically opposite to B.

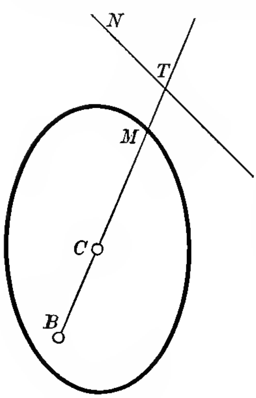


FIG. 50

From equation (**31**), the slope of the tangent is found to be $\frac{b^2z'}{a^2y'}$.

If, then, the point B moves out along a radius CB, z' and y' increase in the same ratio, and consequently the slope is constant; that is to say, if B moves out along a radius, the neutral axis moves parallel to itself.

As z' and y' increase, z and y must decrease, for the products zz' and yy' must be constant in order to satisfy equation (**31**). Therefore the farther B is from the center of gravity, the nearer the corresponding neutral axis is to the center of gravity, and vice versa.

If, in Fig. 50, TN is the neutral axis corresponding to B, it follows, from the above, that $CB \cdot CT$ is a constant wherever B is on the line BT. But if B lies on the ellipse, the corresponding neutral axis is tangent to the ellipse at the point diametrically opposite to B, and in this case the above product becomes $\overline{CM}^2$. Therefore

$$CB \cdot CT = \overline{CM}^2. \tag{33}$$

From this relation, the position of the neutral axis can be determined when the position of the point B is given.

60. Antipole and antipolar. The theorems in the preceding paragraph prove that if the point of application of an eccentric load lies outside, on, or within the inertia ellipse, the corresponding neutral axis cuts this ellipse, is tangent to it, or lies wholly outside it. This relation is analogous to that of poles and polars in analytical geometry, except that in the present case the point and its corresponding line lie on opposite sides of the center instead of on the same side. For this reason the point in the present case is called the **antipole**, and its corresponding line the **antipolar**.

The following theorem is analogous to a well-known theorem of poles and polars.

If the antipole moves along a fixed straight line, the antipolar revolves about a fixed point. Conversely, if the antipolar revolves about a fixed point, the antipole moves along a fixed straight line.

If the antipole moves to infinity, the antipolar, or neutral axis, passes through the center of gravity of the section, which is the ordinary case of pure bending strain. The bending moment in this case can be considered as due to an infinitesimal force at an infinite distance from the center of gravity.

If the antipole coincides with the center of gravity, the neutral axis lies at infinity, which means that the stress is uniformly distributed over the cross section.

Since the stresses on opposite sides of the neutral axis are of opposite sign, if the neutral axis cuts the cross section, stresses of both signs occur (i.e. both tension and compression), whereas if the neutral axis lies outside the cross section, the stress on the section is all of the same sign (i.e. either all tension or all compression).

61. Core section. Let it be required to find all positions of the point of application of an eccentric load such that the stress on the cross section shall all be of the same sign. From the preceding article, the condition for this is that the neutral axis shall not cut the cross section. If, then, all possible lines are drawn touching the cross section or having one point in common with it, and the antipoles of these lines are found, the locus of these antipoles will form a closed figure, called the **core section**.

For a point within or on the boundary of the core section the neutral axis lies entirely without the cross section, or, at most, touches it,

and consequently stress of only one sign occurs. For a point without the core section the corresponding neutral axis cuts the cross section and it is subjected to stresses of both signs.

Problem 90. Construct the core section for a rectangular cross section of breadth b and height h (Fig. 51).

Solution. From Problem 56, $I_z = \frac{bh^3}{12}$, $I_y = \frac{hb^3}{12}$, and the corresponding radii of gyration are $t_z^2 = \frac{I_z}{F} = \frac{h^2}{12}$ and $t_y^2 = \frac{b^2}{12}$. Consequently, the semi-axes of the inertia ellipse are $t_z = \frac{h}{2\sqrt{3}}$ and $t_y = \frac{b}{2\sqrt{3}}$. Having constructed the inertia ellipse, the vertices of the core section will be antipoles of the lines PQ, QR, RS, and SP.

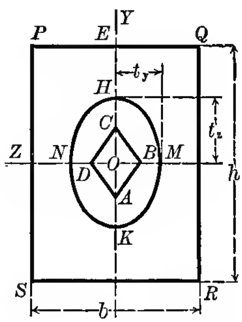


FIG. 51

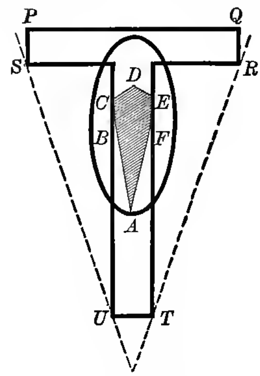


FIG. 52

From Article 59, the antipole of PQ is determined by the relation $OA \cdot OE = \overline{OH}^2$, or, since $OE = \frac{h}{2}$ and $OH = t_z = \frac{h}{2\sqrt{3}}$, $OA = \frac{h}{6}$. Similarly, $OC = \frac{h}{6}$ and $OB = OD = \frac{b}{6}$. Thus the core section is the rhombus $ABCD$, of which the vertices A, B, C, D are the antipoles of the lines PQ, PS, SR, QR respectively, and the sides AB, BC, CD, DA are the antipolars of the points P, S, R, Q respectively.

Problem 91. Construct the core section for the T-shape in Problem 71.

Solution. Six lines can be drawn which will have two or more points in common with the perimeter of the T-shape without crossing it, namely, PQ, QR, RT, TU, US, and SP (Fig. 52). The vertices A, B, C, D, E of the core section are then the antipoles of these six lines respectively.

Problem 92. Construct the core section of the I-beam in Problem 72.

Problem 93. Construct the core section for the channel in Problem 73.

Problem 94. Construct the inertia ellipse and core section for a circular cross section.

62. Application to concrete and masonry construction. Since concrete and masonry are designed to carry only compressive stresses, it

is essential that the point of application of the load shall lie within the core section.

Consider a rectangular cross section of breadth b and height h. For the gravity axes MM and NN (Fig. 53) the corresponding moments of inertia are

$$I_m = \frac{hb^3}{12} \qquad \text{and} \qquad I_n = \frac{bh^3}{12}.$$

Hence the radii of gyration are

$$t_m = \frac{b}{\sqrt{12}} = .2887\,b \qquad \text{and} \qquad t_n = \frac{h}{\sqrt{12}} = .2887\,h,$$

and the inertia ellipse is constructed on these as semi-axes. To determine the core section it is sufficient to find the antipole of each side of the cross section $PQRS$. Suppose A is the antipole of PQ, B the antipole of PS, etc. Then, by Article 60, the antipole of any line through P, such as LL, lies somewhere on AB; that is to say, as the line PQ revolves around P to the position PS, its antipole moves along AB from A to B. The core section in the present case is thus found to be the rhombus $ABCD$.

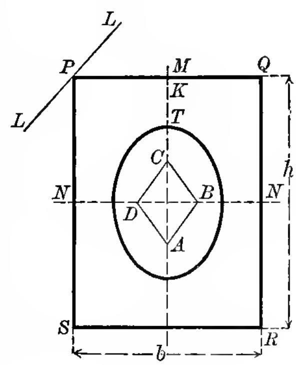


FIG. 53

From Article 59, $OC \cdot OK = \overline{OT}^2 = \frac{h^2}{12}$, since the semi-axes of the ellipse are the radii of gyration. But $OK = \frac{h}{2}$; hence $OC = \frac{h}{6}$ and $AC = \frac{h}{3}$. Similarly, $BD = \frac{b}{3}$. This proves the correctness of the rule ordinarily followed in masonry construction, namely, that in order to insure that the stress shall all be of the same sign, the center of pressure must fall within the middle third of the cross section.

63. Calculation of pure bending strain by means of the core section. Let Fig. 54 represent the cross section of a beam subjected to pure bending strain. In this case the neutral axis passes through the center of gravity of a cross section, and therefore, from Article 60, the strain can be considered as due to an infinitesimal force at an infinite distance from the origin. Under this assumption the stress

due to pure bending strain can be readily calculated by means of the core section, as follows.

Suppose the external bending moment M lies in a plane perpendicular to the plane of the cross section and intersecting it in the line MM. Then, assuming that M is due to an infinitesimal force whose point of application is at an infinite distance from O in the direction OM, the antipolar of this point will be the diameter of the inertia ellipse conjugate to MM. It is proved in analytical geometry that the tangent at the end of a diameter of a conic is parallel to the conjugate diameter. Therefore, if BT is tangent to the inertia ellipse at B, and NN is drawn through O parallel to BT, NN will be the diameter conjugate to MM. Since the greatest stress occurs on the fiber most distant from the neutral axis, the maximum stress will occur at P or R. Through P draw PA parallel to NN and intersecting MM in A. Then, from Article 59,

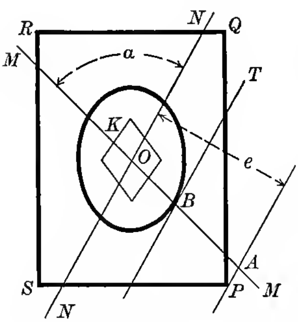


FIG. 54

$$OA \cdot OK = \overline{OB}^2,$$

or, taking the projections of OA, OK, and OB on a line perpendicular to NN,

$$e \cdot OK \sin\alpha = (OB \sin\alpha)^2,$$

where e is the perpendicular distance of PA from O. But $OB \sin\alpha$ is the distance of the tangent BT from O, and, by Article 49, this distance is the radius of gyration t corresponding to the axis NN. Therefore

$$\textbf{(34)} \qquad e \cdot OK \sin\alpha = t^2 = \frac{I_n}{F},$$

where F is the area of the section and I_n is its moment of inertia with respect to NN. The component of the external moment M perpendicular to NN is $M \sin\alpha$. Hence, equating this to the internal moment,

$$\textbf{(35)} \qquad M \sin\alpha = \frac{p_0}{e}\int y(y\,dF) = \frac{p_0}{e}\int y^2 dF = \frac{p_0 I_n}{e},$$

where p_0 is the stress at the distance e from the neutral axis. Substituting in equation (**34**) the value of I_n obtained from equation (**35**),

$$e \cdot OK \sin\alpha = \frac{e \cdot M \sin\alpha}{p_0 F};$$

whence

$$p_0 = \frac{M}{F \cdot OK}. \qquad (36)$$

If, in the handbooks issued by iron and steel companies, the inertia ellipse and core section were drawn on each cross section tabulated, the calculation of the maximum bending stress by formula (**36**) would be extremely simple, requiring merely the measurement of the distance OK.

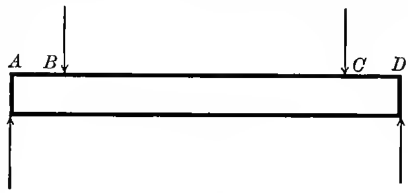


FIG. 55

Problem 95. Calculate the maximum bending stress in Problem 89 by means of the core section.

Solution. The loading is as represented in Fig. 55, in which the portion BC is subjected to pure bending strain. From Problem 89, $M = 20{,}000$ ft. lb. and $F = 48$ in.^{2}. From the diagram of the core section drawn to scale, OK is found to measure .9 in. Therefore, from formula (**36**), $p_0 = 5555$ lb./in.^{2}.

64. Stress trajectories. In Article 27 the principal stresses at any point in a body were defined as the maximum and minimum normal stresses at this point. Lines which everywhere have the direction of the principal stresses are called **stress trajectories.**

In order to determine the stress trajectories, a number of cross sections of the body are taken, and the shear and normal stress calculated for a number of points in each section. The directions which the principal stresses at these points make with the axis of the body can then be found by formula (**6**), Article 26, as explained in Problem 39. The stress trajectories are thus determined as the envelopes of these tangents.

Since the principal stresses at any point are always at right angles, the stress trajectories constitute a family of orthogonal curves.

65. Materials which do not conform to Hooke's law. The preceding articles of this chapter are based on Hooke's law, and consequently the results are applicable only to materials which conform to this law, such as steel, wrought iron, and wood. Other materials, such as cast iron, stone, brick, cement, and concrete, are so lacking in homogeneity that their physical properties are very uncertain, differing not

only for different specimens of the material but also for different portions of the same specimen. For this reason it is impossible to apply to such materials a general method of analysis with any assurance that the results will approximate the actual behavior of the material. For practical purposes, however, the best method is to calculate the strength of such materials by the formulas deduced above, and then modify the result by a factor of safety so large as to include all probable exceptions.

The behavior of cast iron is more uncertain than that of any other material of construction, and it must therefore be used with a larger factor of safety. If two pieces from the same specimen are subjected to tensile strain and to cross-bending strain respectively, it will be found that the ultimate strength deduced from the cross-bending test is about twice as great as that deduced from the tensile test. The reason for this is that the neutral axis does not pass through the center of gravity of a cross section, lying nearer the compression than the tension side, and also because the stresses increase more slowly than their distances from the neutral axis. If, then, it becomes necessary to design a cast-iron beam, the ultimate tensile strength used in the calculation should be that deduced from bending tests.

For materials such as concrete, stone, and cement, the most rational method of procedure is to introduce a correction coefficient η in formula (**18**) and put

$$p = \eta \frac{Me}{I},$$

where it has been found by experiment that for granite $\eta = .96$, for sandstone $\eta = .84$, and for concrete $\eta = .97$.*

66. Design of reënforced concrete beams. Since concrete is a material which does not conform to Hooke's law, and moreover does not obey the same elastic law for tension as for compression, the exact analysis of stress in a plain or reënforced concrete beam would be much more complicated than that obtained under the assumptions of the common theory of flexure. The physical properties of concrete, however, depend so largely on the quality of material and workmanship, that for practical purposes the conditions do not warrant a rigorous analysis. The following simple formulas, although based on

* Föppl, *Festigkeitslehre*, p. 144.

approximate assumptions, give results which agree closely with experiment and practice.

Consider first a plain concrete beam, that is, without reënforcement. The elastic law for tension is in this case (see Fig. 56)

$$\frac{p_t^{m_1}}{s_t} = E_t,$$

and for compression

$$\frac{p_c^{m_2}}{s_c} = E_c.$$

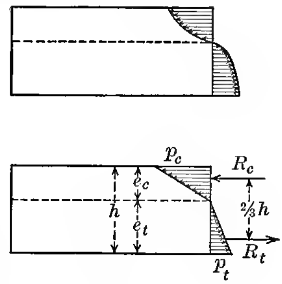


FIG. 56

To simplify the solution, however, assume the straight-line law of distribution of stress, that is, assume $m_1 = m_2 = 1$. Note, however, that this does not make the moduli equal. Assume also that cross sections which were plane before flexure remain plane after flexure (Bernoulli's assumption), which leads to the relation

$$\frac{s_c}{s_t} = \frac{e_c}{e_t},$$

where e_c and e_t denote the distances of the extreme fibers from the neutral axis (Fig. 56).

Now let the ratio of the two moduli be denoted by n, that is, let

$$\frac{E_c}{E_t} = n.$$

Then
$$\frac{p_c}{p_t} = \frac{s_c E_c}{s_t E_t} = n\frac{e_c}{e_t}.$$

For a section of unit width the resultant compressive stress R_c on the section is $R_c = \frac{1}{2} p_c e_c$, and similarly the resultant tensile stress R_t is $R_t = \frac{1}{2} p_t e_t$. Also, since R_c and R_t form a couple, $R_c = R_t$. Hence $p_c e_c = p_t e_t$, or $\frac{p_c}{p_t} = \frac{e_t}{e_c}$; and equating this to the value of the ratio $\frac{p_c}{p_t}$ obtained above, we have

$$e_t = e_c\sqrt{n}.$$

Since the total depth of the beam h is $h = e_c + e_t$, we have, therefore, $e_c = h - e_c\sqrt{n}$, whence

$$e_c = \frac{h}{1+\sqrt{n}};$$

and, similarly, $e_t = h - \frac{e_t}{\sqrt{n}}$, whence

$$e_t = \frac{h}{1+\frac{1}{\sqrt{n}}}.$$

Now, by equating the external moment M to the moment of the stress couple, we have

$$M = (\tfrac{1}{2}\,p_c e_c)\,\tfrac{2}{3}\,h \quad \text{or} \quad M = (\tfrac{1}{2}\,p_t e_t)\,\tfrac{2}{3}\,h,$$

whence, by solving for the unit stresses p_c and p_t,

$$p_c = \frac{3\,M}{h^2}(1+\sqrt{n}), \qquad p_t = \frac{3\,M}{h^2}\left(1+\frac{1}{\sqrt{n}}\right);$$

or, solving one of these two relations for h, say the first, we have

$$h = \sqrt{\frac{3\,M}{p_c}(1+\sqrt{n})}.$$

For ordinary concrete n may be taken as 25. Also, using a factor of safety of 8, the working stress p_c becomes $p_c = 300$ lb./in.^{2} Substituting these numerical values in the above, the formula for the depth of the beam in terms of the external moment takes the simple form

$$h = \frac{\sqrt{M}}{4},$$

h being expressed in inches, and M in inch pounds per inch of width of beam.

Problem 96. A plain concrete slab, supported on two sides only, has a 12-ft. span and carries a load of 200 lb./ft.^{2} Find the required thickness.

Solution. The load is $\frac{200}{144}$ lb./in.^{2}, and hence for a strip 1 in. wide, the maximum moment is $M = \frac{wl^2}{8} = 3600$ in. lb. Consequently the required depth h is $h = \frac{\sqrt{M}}{4} = 15$ in.

For a reënforced concrete beam the tensile strength of the concrete may be neglected. Let E_c and E_s denote the moduli of elasticity for concrete and steel respectively, and let $\frac{E_s}{E_c} = n$. Then if x denotes the distance of the neutral axis from the top fiber (Fig. 57), the assumptions in this case are expressed by the relations

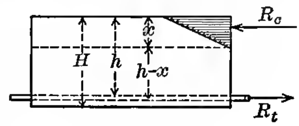


FIG. 57

$$\frac{s_c}{s_s} = \frac{x}{h-x}, \qquad \frac{p_c}{s_c} = E_c, \quad \text{and} \quad \frac{p_s}{s_s} = E_s,$$

whence

$$\frac{s_c}{s_s} = \frac{p_c E_s}{p_s E_c} = n\frac{p_c}{p_s} = \frac{x}{h-x};$$

or, solving for x,

$$x = h\frac{np_c}{p_s + np_c}.$$

Now if F denotes the area of steel reënforcement per unit width of beam, then

$$R_s = p_s F \quad \text{and} \quad R_c = \tfrac{1}{2} p_c x;$$

and consequently, since $R_c = R_s$,

$$\tfrac{1}{2} p_c x = p_s F.$$

Moreover, equating the external moment M to the moment of the stress couple, we have

$$M = \frac{1}{2} p_c x\left(h - \frac{x}{3}\right) \quad \text{or} \quad M = p_s F\left(h - \frac{x}{3}\right).$$

Substituting the value of x in either one of these expressions, say the first, we have

$$M = \frac{1}{2} p_c h \frac{np_c}{p_s + np_c}\left(h - \frac{h}{3}\frac{np_c}{p_s + np_c}\right);$$

whence, solving for h,

$$h = \frac{p_s + np_c}{p_c}\sqrt{\frac{6M}{n(3p_s + 2np_c)}}.$$

For practical work assume $n = 15$, $p_c = 500$ lb./in.^{2} (factor of safety of 5), and $p_s = 15{,}000$ lb./in.^{2} (factor of safety of 4). Substituting these numerical values in the above, the results take the simple form

$$h = .116\sqrt{M}, \qquad h = 3x,$$

$$F = \frac{h}{180}, \qquad x = 60F,$$

$$H = h + \frac{d}{2} + \frac{1}{2},$$

where H denotes the total depth of the beam in inches, d is the diameter of the reënforcement in inches, and M is the external moment in inch pounds per inch of width.

In designing beams by these formulas first find h, then F, and finally H.

Problem 97. A reënforced concrete slab, supported on two sides only, has a 12-ft. span and carries a load of 200 lb./ft.^{2} Find the required thickness of slab and area of metal reënforcement per foot of width.

Solution. As in the preceding example, the maximum moment is $M = 3600$ in. lb. Consequently, $h = .116\sqrt{M} = 6.96$ in.; also $F = \frac{h}{180}$ in.^{2} per inch of width, or $\frac{h \cdot 12}{180}$ in.^{2} per foot of width $= .464$ in.^{2}/ft.; and hence the diameter of the reënforcement is $d = \frac{3}{4}$ in. for round rods spaced one foot apart. Finally, the total depth of slab is $H = 6.96 + \frac{3}{8} + \frac{1}{2} = 7.84$ in., say 8 in.

An interesting application of these formulas is the comparison of the calculated position of the neutral axis in a reënforced concrete beam with that determined experimentally. It has been shown by experiment that when a reënforced concrete beam is loaded, minute cracks appear extending upward from the bottom, showing that practically all the tensile stress is carried by the reënforcement. To render this more obvious, before the concrete is put in, place one or more sheets of pasteboard vertically in the mold in which the beam is made, extending completely across the mold and upward from the bottom to within a distance of the top at least equal to the value of x given by the above formulas. This eliminates entirely the tensile strength of the concrete, which is the assumption upon which the above formulas are based; and when the beam is loaded the extension of the reënforcement causes a crack to appear plainly along the

pasteboard. Since this crack must end at the neutral axis, the position of this axis is thus approximately determined experimentally and may be used to verify the calculated value of x.

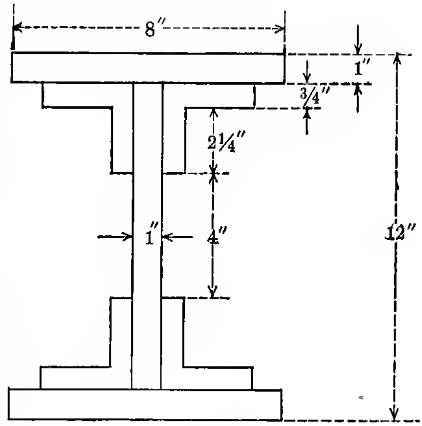


FIG. 58

EXERCISES ON CHAPTER III

Problem 98. A structural steel-built beam is 20 ft. long and has the cross section shown in Fig. 58. Compute its moment of resistance and find the safe uniform load it can carry per linear foot for a factor of safety of 5.

Problem 99. The cast-iron bracket shown in Fig. 59 has at the dangerous section the dimensions shown in the figure. Find the maximum concentrated load it can carry with a factor of safety of 15.

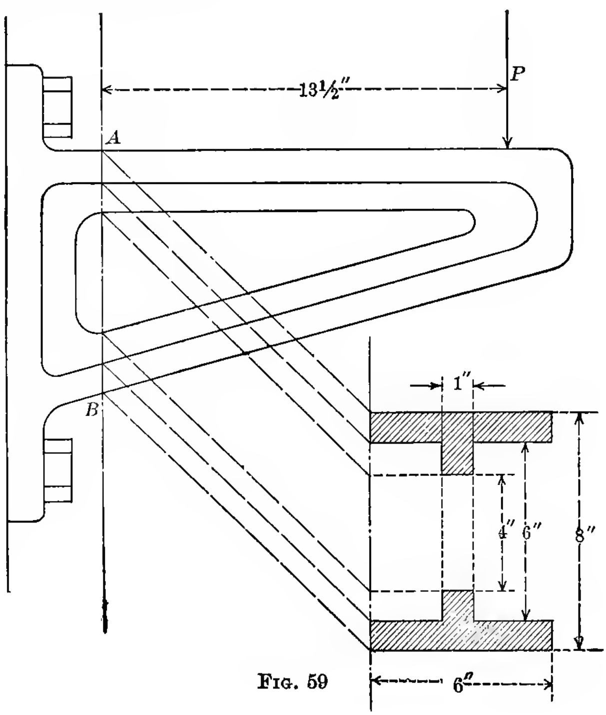


FIG. 59

Problem 100. Find the proper dimensions for a wrought-iron crank shaft of dimensions shown in Fig. 60 for a crank thrust of 1500 lb. and a factor of safety of 6.

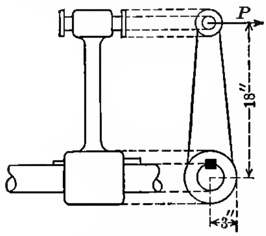


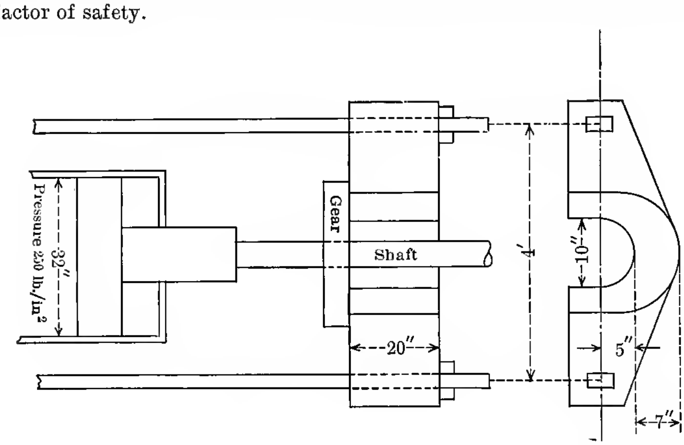
Fig. 60

Problem 101. A wrought-iron pipe 1 in. in external diameter and $\frac{1}{16}$ in. thick projects 6 ft. from a wall. Find the maximum load it can support at the outer end.

Problem 102. The yoke of an hydraulic press used for forcing gears on shafts is of the form and dimensions shown in Fig. 61. The yoke is horizontal with groove up, so that the shaft to be fitted lies in the groove, as shown in plan in the figure. The ram is 32 in. in diameter and under a water pressure of 250 lb./in.² Find the dangerous section of the yoke and the maximum stress at this section.

Problem 103. Design a concrete conduit, 7 ft. square inside, to support a concentrated load of 1000 lb. per linear foot. and determine the size and spacing of the reënforcement.

Problem 104. A 10-in. I-bar weighing 40 lb./ft. is supported on two trestles 15 ft. apart. A chain block carrying a 1-ton load hangs at the center of the beam. Find the factor of safety.



Plan   End Elevation

Fig. 61

Problem 105. The hydraulic punch shown in Fig. 62 is designed to punch a $\frac{5}{8}$-in. hole in a $\frac{5}{8}$-in. plate. The dimensions of the dangerous section AB are as given in the figure. Find the maximum stress at this section.

Problem 106. The load on a car truck is 8 tons, equally distributed between the two wheels (Fig. 63). The axle is of cast steel. Find its diameter for a factor of safety of 15.

Problem 107. The floor of an ordinary dwelling is assumed to carry a load of 50 lb./ft.² and is supported by wooden joists 2 in. by 10 in. in section, spaced 16 in. apart on centers. Find the greatest allowable span for a factor of safety of 10.

Problem 108. A wooden girder supporting the bearing partitions in a dwelling is made up of four 2-in. by 10-in. joists set on edge and spiked together. Find the size of a steel I-beam of equal strength.

Problem 109. A factory floor is assumed to carry a load of 200 lb./ft.^{2} and is supported by steel I-beams of 16 ft. span and spaced 4 ft. apart on centers. What size I-beam is required for a factor of safety of 4?

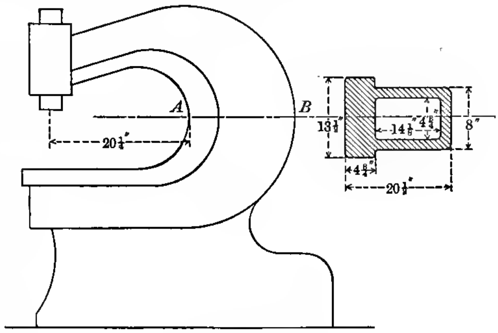


FIG. 62

Problem 110. Find the required size of a square wooden beam of 14 ft. span to carry an axial tension of 2 tons and a uniform load of 100 lb./ft.

Problem 111. A reënforced concrete beam 10 in. wide and 22 in. deep has four $1\frac{1}{4}$-in. round bars with centers 2 in. above the lower face. The span is 16 ft. The beam is simply supported at the ends. Find the safe load per linear foot for a working stress in the concrete of 500 lb./in.^{2}, and also find the tensile stress in the reënforcement.

Problem 112. A reënforced concrete floor is to carry a load of 200 lb./ft.^{2} over a span of 14 ft. Find the required thickness of the slab and area of the reënforcement for working stresses of 500 lb./in.^{2} in the concrete and 15,000 lb./in.^{2} in the reënforcement.

Problem 113. A reënforced concrete beam of 16 ft. span is 18 in. deep, 9 in. wide, and has to support a uniform load of 1000 lb. per linear foot. Determine the amount of steel reënforcement required, bars to have centers 2 in. above lower face of beam.

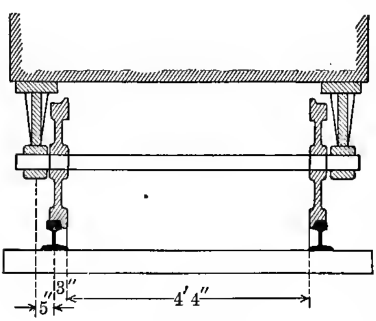


FIG. 63

Problem 114. Find the maximum moment and maximum shear, and sketch the shear and moment diagrams for a cantilever beam 8 ft. long, weighing 20 lb./ft., with concentrated loads of 200 and 300 lb. at 3 and 5 ft. respectively from the free end.

Problem 115. Find the maximum moment and maximum shear, and sketch the shear and moment diagrams for a cantilever beam 12 ft. long, carrying a total uniform load of 50 lb./ft. and concentrated loads of 200, 150, and 400 lb. at distances of 2, 4, and 7 ft. respectively from the fixed end.

Problem 116. A beam 30 ft. long carries concentrated loads of 1 ton at the left end, 1.5 tons at the center, and 2 tons at the right end, and rests on two supports, one 4 ft. from the left end and the other 6 ft. from the right end. Sketch the shear and moment diagrams and find the maximum shear and maximum moment.

Problem 117. A beam 20 ft. long bears a uniform load of 100 lb. per linear foot and rests on two supports 10 ft. apart and 5 ft. from the ends of the beam. Find the maximum moment and shear, and sketch the shear and moment diagrams.

Problem 118. Find the maximum moment and maximum shear, and sketch the shear and moment diagrams for a simple beam 10 ft. long, bearing a total uniform load of 100 lb. per linear foot and concentrated loads of 1 ton at 4 ft. from the left end and 2 tons at 3 ft. from the right end.

Note on Shear and Moment Diagrams

It is important to be able to sketch readily by inspection the shear and moment diagrams corresponding to any given loading. To acquire this ability it is only necessary to observe the characteristic features of such diagrams. The more important of these are as follows:

The slope of the moment curve is equal to the shear. From this, the following conclusions are obtainable.

Where the moment is a maximum the shear is zero. Note, however, that for concentrated loads the moment has no calculus maximum. In this case, where the moment has its greatest value, the shear passes through zero because the slope of the moment diagram necessarily changes from positive to negative at this point.

Where the moment is constant the shear is zero.

For a uniform load the moment diagram is a parabola and the shear diagram is an inclined line whose slope is equal to the load per unit of length. Mathematically this means that the parabola is a curve whose slope changes uniformly from point to point.

For concentrated loads the moment diagram is a broken straight line, and the shear diagram is a series of horizontal lines or steps.

For uniform and concentrated loads combined, the moment diagram is a series of parabolic arcs, and the shear diagram is a series of inclined lines or sloping steps.

At the ends of a simple beam the moment is always zero.

Where the moment diagram crosses the axis, the elastic curve or center line of the beam has a point of inflection; that is to say, the beam is curved upward on one side of this point and curved downward on the other side. Such a point is called a *point of contraflexure*. The tensile stress changes from the bottom to the top on opposite sides of a point of contraflexure, and such points are therefore of especial importance in the case of reënforced concrete beams, as the reënforcement must always follow the tensile stress.

The area subtended by the shear diagram up to any point is equal to the moment at this point, since $\frac{dM}{dx} = Q$ and therefore $M = \int Qdx$.

CHAPTER IV

FLEXURE OF BEAMS

67. Elastic curve. If a beam is subjected to transverse loading, its axis is bent into a curve called the **elastic curve**. The differential equation of the elastic curve is found as follows.

Let $ABDE$ (Fig. 64) represent a portion of a bent beam limited by two adjacent cross sections AB and DE, and let C be a point in the intersection of these two cross sections. Then C is the center of curvature of the elastic curve FH. Let $d\beta$ denote the angle ACE, and through H draw LK parallel to AC;

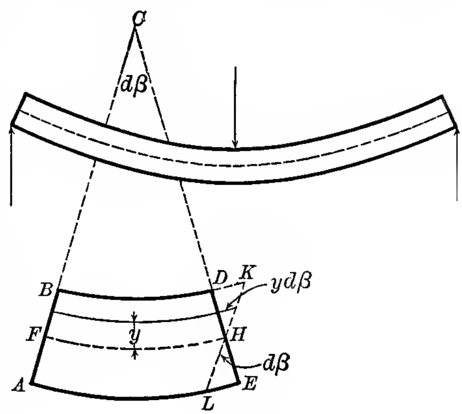


FIG. 64

then the angle LHE is also equal to $d\beta$. Since the normal stress is zero at the neutral axis, the fiber FH is unchanged in length by the strain. Therefore, from Fig. 64, the change in length of a fiber at a distance y from the elastic curve is $yd\beta$, where $d\beta$ is expressed in circular measure. Consequently, the unit deformation of such a fiber is

$$s = \frac{yd\beta}{dx}.$$

By Hooke's law, $\frac{p}{s} = E$ where $p = \frac{My}{I}$; hence

$$\frac{My}{Is} = E.$$

Inserting in this expression the value of s just found, $\frac{Mydx}{Iyd\beta} = E$; whence

$$d\beta = \frac{Mdx}{EI}.$$

Let the radius of curvature CF of the elastic curve be denoted by ρ. Then $\rho d\beta = dx$, and inserting this value of $d\beta$ in the above equation, it becomes

$$\rho = \frac{EI}{M}.$$

From the differential calculus, the radius of curvature of any curve can be expressed by the formula

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

But since the deformation of the beam is assumed to be small, the slope of the tangent at any point of the elastic curve is small; that is to say, $\frac{dy}{dx}$ is infinitesimal, and consequently $\left(\frac{dy}{dx}\right)^2$ can be neglected in comparison with $\frac{d^2y}{dx^2}$. Under this assumption $\rho = \frac{1}{\frac{d^2y}{dx^2}}$, and therefore $\frac{EI}{M} = \rho = \frac{1}{\frac{d^2y}{dx^2}}$; whence

$$\text{(37)} \qquad \boldsymbol{EI\frac{d^2y}{dx^2} = M,}$$

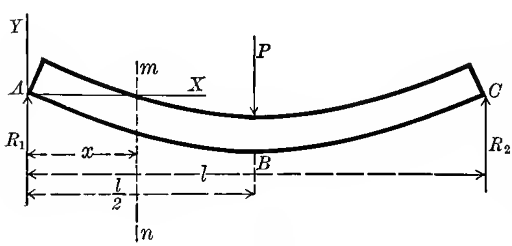


Fig. 65

which is the required differential equation of the elastic curve.

In what follows the external bending moment M is assumed to be negative if it tends to revolve the portion of the beam under consideration in a clockwise direction, and positive if the revolution is counter-clockwise.

Problem 119. Find the equation of the elastic curve and the deflection at the center of a simple beam of length l, bearing a single concentrated load P at its center.

Solution. The elastic curve in this case consists of two branches, AB and BC (Fig. 65).

Consider the portion of the beam on the left of any section mn, distant x from the left support. Then $M = -R_1x = -\frac{P}{2}x$, and consequently the differential equation of the branch AB of the elastic curve is

$$EI\frac{d^2y}{dx^2} = -\frac{Px}{2}.$$

Integrating twice,

$$EI\frac{dy}{dx} = -\frac{Px^2}{4} + C_1,$$

and

$$EIy = -\frac{Px^3}{12} + C_1x + C_2.$$

At B, $x = \frac{l}{2}$ and $\frac{dy}{dx} = 0$, since the tangent at B is horizontal. Substituting these values in the first integral, $C_1 = \frac{Pl^2}{16}$. At A, $x = 0$ and $y = 0$; hence $C_2 = 0$. Consequently, the equation of the left half of the elastic curve is

$$y = \frac{Px}{48\,EI}(3\,l^2 - 4\,x^2).$$

The deflection D at the center is the value of y for $x = \frac{l}{2}$; hence

$$D = \frac{Pl^3}{48\,EI}.$$

Problem 120. Find the equation of the elastic curve and the maximum deflection for a cantilever of length l, bearing a single concentrated load P at the end.

Problem 121. Find the equation of the elastic curve and the maximum deflection for a simple beam of length l, bearing a single concentrated load P at a distance d from the left support.

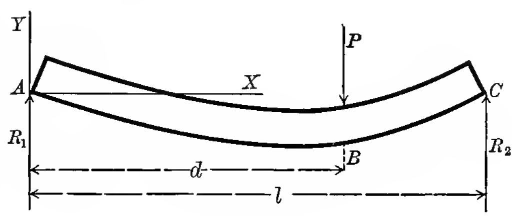


FIG. 66

Solution. The elastic curve in this case consists of two branches, AB and BC (Fig. 66). For a point in AB distant x from the left support, $M = \frac{-P(l-d)x}{l}$. Therefore

$$EI\frac{d^2y}{dx^2} = -\frac{P(l-d)x}{l}.$$

Integrating twice,

$$EI\frac{dy}{dx} = -\frac{P(l-d)x^2}{2\,l} + C_1,$$

and $$EIy = -\frac{P(l-d)x^3}{6l} + C_1 x + C_2.$$

At A, $x = 0$ and $y = 0$; therefore $C_2 = 0$. In order to determine C_1 it is necessary to find the equation of BC.

Taking a section on the right of B, $M = -\frac{Pd(l-x)}{l}$, and consequently

$$EI\frac{d^2y}{dx^2} = -\frac{Pd(l-x)}{l}.$$

Integrating twice,

$$EI\frac{dy}{dx} = -\frac{Pd}{l}\left(lx - \frac{x^2}{2}\right) + C_3,$$

and $$EIy = -\frac{Pd}{l}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_3 x + C_4.$$

At C, $x = l$ and $y = 0$; therefore $C_4 = \frac{Pdl^2}{3} - C_3 l$.

Now at B both branches of the elastic curve have the same ordinate and the same slope. Therefore, putting $x = d$ in the above integrals and equating the slopes and ordinates of the two branches,

$$\frac{-P(l-d)d^2}{2l} + C_1 = -\frac{Pd}{l}\left(ld - \frac{d^2}{2}\right) + C_3,$$

$$\frac{-P(l-d)d^3}{6l} + C_1 d = -\frac{Pd}{l}\left(\frac{ld^2}{2} - \frac{d^3}{6}\right) + C_3 d + \frac{Pdl^2}{3} - C_3 l.$$

Solving these two equations simultaneously for C_1 and C_3,

$$C_1 = \frac{Pd}{6l}(2l-d)(l-d),$$

$$C_3 = \frac{Pd}{6l}(2l^2 + d^2).$$

Substituting these values of C_1 and C_3 in the above integrals, the equation of the branch AB becomes, after reduction,

$$y = \frac{Px(l-d)}{6EIl}(2ld - d^2 - x^2),$$

and the equation of BC becomes

$$y = \frac{Pd(l-x)}{6EIl}(2lx - d^2 - x^2).$$

Since the load is not at the center of the beam, the maximum deflection will occur in the longer segment. Moreover, at the point of maximum deflection the tangent is horizontal, that is, $\frac{dy}{dx} = 0$. Therefore, equating to zero the first differential coefficient of the branch AB,

$$0 = -\frac{P(l-d)x^2}{2l} + \frac{Pd}{6l}(2l-d)(l-d);$$

from which the distance of the point of maximum deflection from the left support is found to be

$$x = \sqrt{\frac{d(2l-d)}{3}},$$

and the deflection at this point is

$$D = \frac{P(l-d)}{3EIl}\left[\frac{d(2l-d)}{3}\right]^{\frac{3}{2}}.$$

Problem 122. Find the equation of the elastic curve and the maximum deflection for a simple beam of length l, bearing a uniform load of w lb. per unit of length.

Problem 123. Find the equation of the elastic curve and the maximum deflection for a cantilever of length l, uniformly loaded with a load of w lb. per unit of length.

Problem 124. A Carnegie I-beam, No. B 13, is 10 ft. long and bears a load of 25 tons at its center. Find the deflection of the point of application of the load.

NOTE. From the Carnegie handbook, the moment of inertia of the beam about a neutral axis perpendicular to the web is $I = 84.9$ in.^{4}

Problem 125. Find the deflection of the beam in the preceding problem at a point 4 ft. from one end.

68. Limitation to Bernoulli's assumption. In Article 39 it was stated that Bernoulli's assumption formed the basis of the common

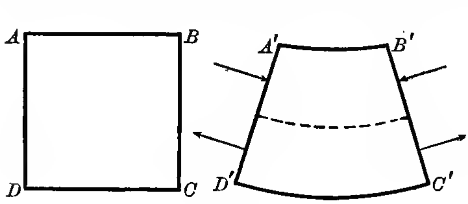


FIG. 67

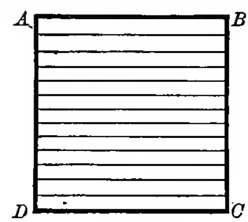


FIG. 68

theory of flexure. In the case of a prismatic beam subjected to pure bending strain, this assumption is rigorously correct. For if the opposite faces of a prism $ABCD$ (Fig. 67) are acted upon by equal bending moments of opposite sign, both faces must, by reason of symmetry, remain plane and take a position such as $A'B'C'D'$ in the figure.

However, if shearing stress also occurs, Bernoulli's assumption is no longer absolutely correct. In Article 55 it was proved that the distribution of shear over any cross section limited by parallel sides varies as the ordinates to a parabola. Consequently, if the beam is supposed cut into thin layers by horizontal planes, as represented in

Fig. 68, the shear will tend to slide these layers one upon another. By Hooke's law, the amount of this sliding for different layers will also vary as the ordinates to a parabola, being zero at top and bottom and a maximum at the center. Therefore, if the elongations and contractions of the fibers due to bending stress are combined with the sliding due to shear, the resultant deformation of the prism will be as represented in Fig. 69.

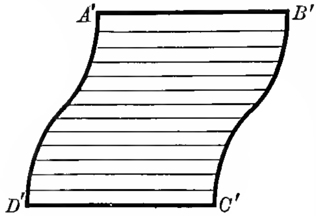


FIG. 69

69. Effect of shear on the elastic curve. In addition to the horizontal shearing stress acting at any point in a beam, there is a shearing stress of equal intensity acting in a vertical direction. The effect of this vertical shear is to slide each cross section past its adjacent cross section, as represented in Fig. 70, and thus increase the deflection of the beam.

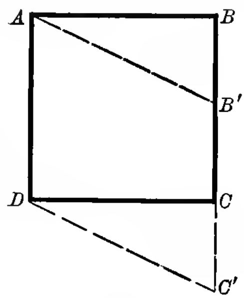


FIG. 70

In Article 83 a general formula is derived by means of which the shearing deflection can be calculated in any given case. It is found, however, that in all ordinary cases the shearing deflection is so small that it can be neglected, in comparison with the deflection due to bending strain. The point to be remembered, then, is that the shearing deflection is negligible but not zero.

In precise laboratory experiments for the determination of Young's modulus it should always be ascertained whether or not the shearing deformation can be neglected without affecting the precision of the result.

70. Built-in beams. If the ends of a beam are secured in such a way as to be immovable, the beam is said to be **built-in**. Examples of built-in beams are found in reënforced concrete construction, in which all parts are monolithic. Thus a floor beam in a building constructed of reënforced concrete is of one piece with its supporting girders, and consequently its ends are immovable.

Since the tangents at the ends of a built-in beam are horizontal, $\frac{dy}{dx} = 0$ at these points. Also, from Fig. 71, it is obvious that the

elastic curve of a built-in beam differs from that for a simple beam in having two points of inflection, A and B. At these points the curvature is zero, that is, $\frac{d^2y}{dx^2} = 0$, and consequently the bending moment is also zero, since $EI\frac{d^2y}{dx^2} = M$.

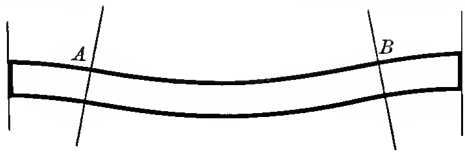


FIG. 71

Problem 126. Find the equation of the elastic curve and the maximum deflection for a beam of length l, fixed at both ends and bearing a uniform load of w lb. per unit of length.

Solution. Let M_a and M_b denote the moments at the supports (Fig. 72). The vertical reactions at the supports are each equal to $\frac{wl}{2}\cdot$

Consequently, the bending moment at a point distant x from the left support is

$$M_x = M_a - \frac{wlx}{2} + \frac{wx^2}{2},$$

and therefore

$$EI\frac{d^2y}{dx^2} = M_a - \frac{wlx}{2} + \frac{wx^2}{2}.$$

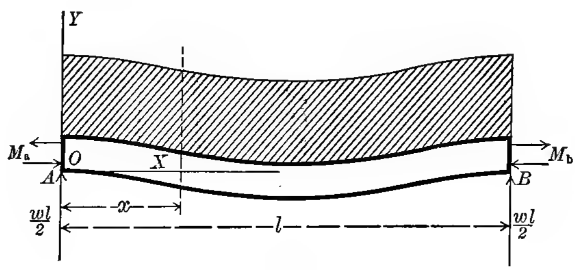


FIG. 72

Integrating,

$$EI\frac{dy}{dx} = M_a x - \frac{wlx^2}{4} + \frac{wx^3}{6} + C_1.$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$; therefore $C_1 = 0$. At B, $x = l$ and $\frac{dy}{dx} = 0$; therefore $M_a = \frac{wl^2}{12}\cdot$ Substituting this value of M_a in the above integral, and integrating again,

$$EIy = \frac{wl^2x^2}{24} - \frac{wlx^3}{12} + \frac{wx^4}{24} + C_2.$$

At A, $x = 0$ and $y = 0$; therefore $C_2 = 0$. Consequently, the equation of the elastic curve is, after reduction,

$$y = \frac{wx^2(l-x)^2}{24\,EI}.$$

Putting $x = \frac{l}{2}$ in this equation, the maximum deflection is found to be

$$D = \frac{wl^4}{384\,EI}.$$

At the points of inflection $\frac{d^2y}{dx^2} = 0$. Therefore

$$0 = M_a - \frac{wlx}{2} + \frac{wx^2}{2};$$

whence

$$x = \frac{l}{2} \pm \frac{l}{\sqrt{12}} = .212\,l \text{ or } .788\,l,$$

which are the distances of the two points of inflection from the left support.

Problem 127. A beam of length l is fixed at both ends and bears a single concentrated load P at a distance d from the left end. Find the deflection at the point of application of the load.

Problem 128. From the result of Problem 127, find the deflection at the point of application of the load when the load is at the center.

Problem 129. A concrete girder 16 ft. long, 18 in. deep, and 12 in. wide is reënforced by two 1-in. twisted square steel rods near its lower face, and bears a uniform load of 250 lb. per linear inch. The moment of inertia of the equivalent homogeneous section about its neutral axis (Article 49) is found to be $I_c = 7230$ in.^{4} Find the maximum deflection.

71. Continuous beams. A continuous beam is one which is supported at several points of its length, and thus extends continuously over several openings. If the reactions of the several supports were known, the distribution of stress in the beam and the equation of the elastic curve could be found by the methods employed in the preceding articles. The first step, therefore, is to determine the unknown reactions. General methods for determining these will be explained in Articles 72, 78, 80, and 81. The two following problems illustrate special methods of treating the two simple cases considered.

Problem 130. A beam is simply supported at its center and ends, and bears a single concentrated load P at the center of each span. Assuming that the supports are at the same level, find their reactions and the equation of the elastic curve.

Solution. Let each span be of length l, and assume the origin of coördinates at O (Fig. 73). Consider the portion of the beam on the right of a section mn, distant x from O. Then, if $x < \frac{l}{2}$,

$$EI\frac{d^2y}{dx^2} = P\left(\frac{l}{2} - x\right) - R_3(l - x).$$

Integrating twice,

$$EI\frac{dy}{dx} = P\left(\frac{lx}{2} - \frac{x^2}{2}\right) - R_3\left(lx - \frac{x^2}{2}\right) + C_1, \tag{38}$$

$$EIy = P\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - R_3\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_1x + C_2. \tag{39}$$

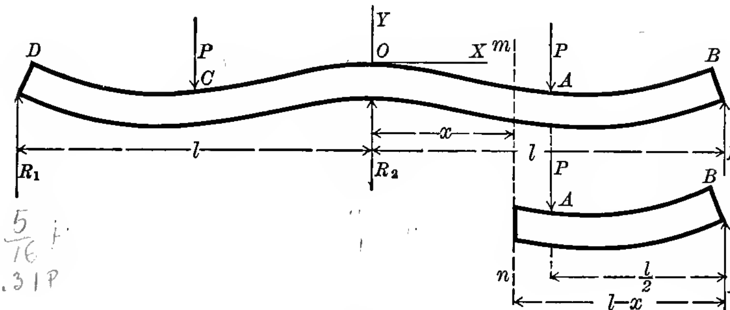
At O, $x = 0$ and $\dfrac{dy}{dx} = 0$; therefore $C_1 = 0$. Also at O, $x = 0$ and $y = 0$; therefore $C_2 = 0$.

Let x be greater than $\dfrac{l}{2}$. Then the differential equation of the branch AB becomes

$$EI\frac{d^2y}{dx^2} = -R_3(l - x).$$

Integrating,

$$EI\frac{dy}{dx} = -R_3\left(lx - \frac{x^2}{2}\right) + C_3. \tag{40}$$



5/16 P

5/16 P
.31 P
c .5

FIG. 73

At A both branches, OA and AB, have the same slope. Therefore, putting $x = \dfrac{l}{2}$ in (**38**) and (**40**), and equating the values of $\dfrac{dy}{dx}$ thus obtained,

$$P\left(\frac{l^2}{4} - \frac{l^2}{8}\right) - R_3\left(\frac{l^2}{2} - \frac{l^2}{8}\right) = -R_3\left(\frac{l^2}{2} - \frac{l^2}{8}\right) + C_3;$$

whence

$$C_3 = \frac{Pl^2}{8}.$$

Substituting this value of C_3 in equation (**40**), and integrating again,

$$EIy = -R_3\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \frac{Pl^2x}{8} + C_4. \tag{41}$$

At A both curves have the same ordinate. Therefore, putting $x = \dfrac{l}{2}$ in equations (**39**) and (**41**), and equating the values of y thus obtained,

$$P\left(\frac{l^3}{16} - \frac{l^3}{48}\right) - R_3\left(\frac{l^3}{8} - \frac{l^3}{48}\right) = -R_3\left(\frac{l^3}{8} - \frac{l^3}{48}\right) + \frac{Pl^3}{16} + C_4;$$

whence

$$C_4 = -\frac{Pl^3}{48}.$$

The equations of both branches of the elastic curve are now determined except that the reaction R_3 is still unknown. Since B is assumed to be on the same level with O, its ordinate is zero. Therefore, to determine R_3, put $x = l$ and $y = 0$ in equation (**41**); whence

$$R_3 = \tfrac{5}{16} P.$$

From symmetry $R_1 = R_3$. Therefore

$$R_2 = 2P - (R_1 + R_3) = \tfrac{11}{8} P.$$

Problem 131. Determine the reactions of the supports for a beam simply supported at its center and ends, and bearing a uniform load of w lb. per unit of length.

Solution. If the end supports were removed, the beam would consist of two cantilevers, AB and BC (Fig. 74), each of length l and bearing a uniform load.

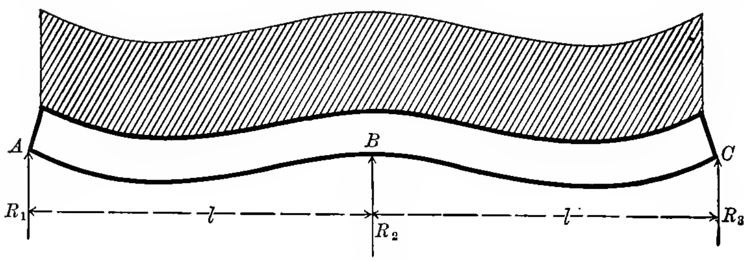


FIG. 74

From Problem 123, the deflection at the end of such a beam is $D = \frac{wl^4}{8EI}$. But the reaction R_3 (or R_1) must be of such amount as to counteract this deflection; and, from Problem 120, the deflection at the end of a cantilever bearing a single concentrated load R_3 is $D = \frac{R_3 l^3}{3EI}$. Therefore

$$\frac{R_3 l^3}{3EI} = \frac{wl^4}{8EI};$$

whence

$$R_3 = \tfrac{3}{8} wl.$$

From symmetry, $R_1 = R_3$. Consequently,

$$R_2 = 2wl - (R_1 + R_3) = \tfrac{5}{4} wl.$$

Having found the reactions of the supports, the equations of the elastic curves can be determined as in the preceding problems.

72. Theorem of three moments. The theorem of three moments is an algebraic relation between the bending moments at three consecutive piers of a continuous beam. The theorem is due to **Clapeyron,**

and first appeared in the *Comptes Rendus* for December, 1857. The following is a simplified proof of the theorem for the case of uniform loading.

Let A, B, C be three consecutive piers of a continuous beam at the same height, and let M_a, M_b, M_c and R_a, R_b, R_c denote the bending moments and reactions at these three points respectively (Fig. 75). Also let l_1, l_2 denote the lengths of the two spans considered, w_1, w_2 the unit loads on them, and Q'_a, Q''_a the shears on the left and right of R_a respectively, with a similar notation for the other supports. Then, taking A as origin, the differential equation of AB is

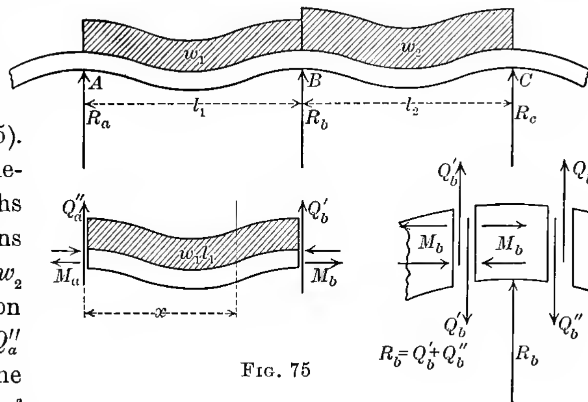


FIG. 75

$$(42) \qquad EI\frac{d^2y}{dx^2} = M_a + Q''_a x - \frac{w_1 x^2}{2}.$$

Integrating twice,

$$(43) \qquad EI\frac{dy}{dx} = M_a x + Q''_a \frac{x^2}{2} - \frac{w_1 x^3}{6} + C_1,$$

and

$$EIy = M_a \frac{x^2}{2} + Q''_a \frac{x^3}{6} - \frac{w_1 x^4}{24} + C_1 x + C_2.$$

At A, $x = 0$ and $y = 0$; hence $C_2 = 0$. At B, $x = l_1$ and $y = 0$; hence

$$C_1 = -\frac{1}{2} M_a l_1 - \frac{1}{6} Q''_a l_1^2 + \frac{w_1 l_1^3}{24}.$$

In equation (**42**), if $x = l_1$, $EI\dfrac{d^2y}{dx^2} = M_b$. Therefore

$$(44) \qquad M_b = M_a + Q''_a l_1 - \frac{w_1 l_1^2}{2}.$$

If $\left(\dfrac{dy}{dx}\right)_b$ denotes the slope of the elastic curve AB at B, then, from equation (**43**),

$$(45) \qquad EI\left(\frac{dy}{dx}\right)_b = \frac{1}{2} M_a l_1 + \frac{1}{3} Q''_a l_1^2 - \frac{w_1}{8} l_1^3.$$

Similarly, by taking the origin at C and reckoning backward toward B, it will be found that

$$(46) \qquad M_b = M_c + Q'_c l_2 - \frac{w_2 l_2^2}{2},$$

and

$$(47) \qquad -EI\left(\frac{dy}{dx}\right)_b = \frac{1}{2} M_c l_2 + \frac{1}{3} Q'_c l_2^2 - \frac{w_2 l_2^3}{8}.$$

Equating the values of $\left(\frac{dy}{dx}\right)_b$ from equations (**45**) and (**47**), and eliminating Q''_a and Q'_c from the resulting equation by means of equations (**44**) and (**46**),

$$\frac{1}{2} M_a l_1 + \left(\frac{1}{3} M_b l_1 - \frac{1}{3} M_a l_1 + \frac{w_1 l_1^3}{6}\right) - \frac{w_1 l_1^3}{8}$$
$$= -\frac{1}{2} M_c l_2 - \left(\frac{1}{3} M_b l_2 - \frac{1}{3} M_c l_2 + \frac{w_2 l_2^3}{6}\right) + \frac{w_2 l_2^3}{8};$$

whence

$$M_a l_1 + 2 M_b (l_1 + l_2) + M_c l_2 = -\frac{w_1 l_1^3 + w_2 l_2^3}{4},$$

which is the required **theorem of three moments.**

If the beam extends over n supports, this theorem furnishes $n - 2$ equations between the n moments at the supports, the remaining two equations necessary for solution being furnished by the terminal conditions at the ends of the beam.

Problem 132. A continuous beam of two equal spans bears a uniform load extending continuously over both spans. Find the bending moments and reactions at the supports.

Solution. In the present case $w_1 = w_2 = w$, $l_1 = l_2 = l$, and $M_a = M_c = 0$. Consequently, the theorem reduces to

$$2 M_b (2 l) = -\frac{2 w l^3}{4};$$

whence

$$M_b = -\frac{w l^2}{8}.$$

From equation (**44**),

$$\frac{-w l^2}{8} = Q''_a l - \frac{w l^2}{2};$$

whence

$$Q''_a = R_a = \tfrac{3}{8} w l.$$

From symmetry, $R_a = R_c$, and consequently

$$R_b = \tfrac{5}{4}\, wl.$$

Problem 133. A continuous beam of four equal spans is uniformly loaded. Find the bending moments and reactions at the supports.

Solution. The system of simultaneous equations to be solved in this case is

$$M_1 = M_5 = 0,$$
$$M_1 + 4\,M_2 + M_3 = -\frac{wl^2}{2},$$
$$M_2 + 4\,M_3 + M_4 = -\frac{wl^2}{2},$$
$$M_3 + 4\,M_4 + M_5 = -\frac{wl^2}{2},$$

the solution of which gives

$$M_2 = M_4 = -\tfrac{3}{28}\,wl^2, \qquad M_3 = -\tfrac{1}{14}\,wl^2,$$
$$Q_1'' = \tfrac{11}{28}\,wl, \quad Q_2' = \tfrac{17}{28}\,wl, \quad Q_2'' = \tfrac{15}{28}\,wl, \quad Q_3' = Q_3'' = \tfrac{13}{28}\,wl,$$
$$R_1 = R_5 = Q_1'' = \tfrac{11}{28}\,wl, \quad R_2 = R_4 = Q_2' + Q_2'' = \tfrac{8}{7}\,wl, \quad R_3 = Q_3' + Q_3'' = \tfrac{13}{14}\,wl.$$

Problem 134. A continuous beam of five equal spans is uniformly loaded. Find the moments and reactions at the supports.

73. Work of deformation. In changing the shape of a body the points of application of the external forces necessarily move, and therefore do a certain amount of work called the **work of deformation.**

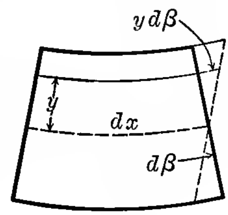


FIG. 76

To find the amount of this work of deformation for a prismatic beam, consider two adjacent cross sections of the beam at a distance dx apart (Fig. 76). Suppose one of these cross sections remains stationary and the other turns through an angle $d\beta$ with reference to the first. Then the change in length of a fiber at a distance y from the neutral axis is $yd\beta$, and therefore, by Hooke's law,

$$\frac{yd\beta}{dx} = \frac{p}{E},$$

where p is the intensity of the stress on the fiber. By the straight-line law, $p = \frac{My}{I}$, and hence

$$d\beta = \frac{Mdx}{EI}.$$

Since one of the cross sections is assumed to be stationary, the stress acting on it does no work. On the other cross section the normal

stress forms a moment equal to M. This moment is zero when first applied, and gradually increases to its full value, its average value being $\frac{1}{2}M$. Therefore the work done by the normal stress on this cross section is

$$dW = \frac{1}{2}Md\beta = \frac{M^2dx}{2\,EI}.$$

Hence the total work of deformation for the entire beam is

$$W = \frac{1}{2}\int\frac{M^2dx}{EI}.$$

Problem 135. As an application of the above, find the deflection at the center of a simple beam of length l, bearing a single concentrated load P at the center.

Solution. Let D denote the deflection at the center. Then the external work of deformation is

$$W = \tfrac{1}{2}PD.$$

At a point distant x from the left support the bending moment is $M = \frac{Px}{2}$, and consequently the internal work of deformation is

$$W = \int_0^{\frac{l}{2}}\frac{\left(\frac{Px}{2}\right)^2dx}{EI} = \frac{P^2l^3}{96\,EI}.$$

Therefore $\frac{1}{2}PD = \frac{P^2l^3}{96\,EI}$; whence $D = \frac{Pl^3}{48\,EI}$.

Problem 136. Find the internal work of deformation for a rectangular wooden beam 10 ft. long, 10 in. deep, and 8 in. wide, which bears a uniform load of 250 lb. per foot of length.

74. Impact and resilience. If the stress lies within the elastic limit of the material, the body returns to its original shape upon removal of the external forces, and the internal work of deformation is given out again in the form of mechanical energy. The internal work of deformation is thus a form of potential energy, and from this point of view is called **resilience.** The work done in straining a unit volume of a material to the elastic limit is called the **modulus of elastic resilience** of the material.

It is therefore represented by the area under the strain curve up to the elastic limit, or, expressed as a formula,

$$\textbf{Mod. elas. resilience} = \frac{\textbf{(stress at elastic limit)}^2}{\textbf{2 modulus of elasticity}}.$$

When a load is suddenly applied to a beam, as when a body falls on the beam, or in the case of a railway train passing quickly over a girder, the deflection of the beam is much greater than it would be if the load was applied gradually, for in this case the full amount of the load is applied at the start instead of gradually increasing from zero up to this amount. Since the load is not sufficiently great to cause the beam to retain this deflection, the resilience of the beam causes it to vibrate back and forth until the effect of the shock dies away. The sudden application of a load is called **impact**, and the study of its effect is of especial importance in designing machines, railway bridges, or any construction liable to shocks.

If a simple beam deflects an amount D under a load P suddenly applied, the work of deformation is PD. If the beam deflects the same amount under a load P' gradually applied, the work of deformation is $\frac{1}{2} P'D$. Hence

$$P' = 2P.$$

In other words, the strain produced in a beam by a load applied suddenly is equivalent to the strain produced by a load twice as great applied gradually. In practical work P' is assumed to be about $\frac{3}{2} P$ instead of $2P$, for it is impossible to apply a load instantaneously at the most dangerous section.

If a body of weight P falls on a beam from a height h and produces a deflection D, the work done by P is $P(h + D)$. Therefore, if P' is the amount of a static load which would produce the same deflection,

$$\frac{1}{2} P'D = P(h + D).$$

In order to find P' from this equation D must be expressed in terms of P' and its value substituted in the above expression before solving for P'.

Problem 137. A Cambria steel I-beam, No. B 33, is 12 ft. long and 10 in. deep, and has a moment of inertia about an axis perpendicular to the web of 122.1 in.^{4}. What is the maximum load that can fall on the center of the beam from a height of 6 in. without producing a stress greater than 25,000 lb./in.^{2}, if 75 per cent of the kinetic energy of the falling body is transformed into work of deformation?

Solution. Let P denote the weight of the falling body and P' the amount of a static load which would produce the same work of deformation. Then, since the moment at the center of the beam is $M = \frac{P'l}{4}$, $p = \frac{Me}{I} = \frac{P'le}{4I}$, whence $P' = \frac{4pI}{le}$.

The deflection of a beam bearing a static load P' at the center is $D = \frac{P'l^3}{48\,EI}$ (Problem 119), or, substituting in this the value of P', $D = \frac{pl^2}{12\,Ee}$. Assuming $E = 30{,}000{,}000$ lb./in.², and replacing p, l, and e by the values given in the problem,

$$D = .288 \text{ in.}$$

Consequently, the work of deformation is

$$W = \frac{1}{2}P'D = \frac{p^2Il}{6\,Ee^2} = 2442 \text{ in. lb.}$$

Therefore, from the equality $\frac{1}{2}P'D = P(h + D)$, we have

$$2442 = .75\,P(6 + .288);$$

whence

$$P = 518 \text{ lb.}$$

Problem 138. From what height can a weight of half a ton fall on the middle of the beam in the preceding problem without producing a stress greater than 40,000 lb./in.²?

75.* Influence line for bending moment. As a load moves over a structure the bending moment and shear at any given point change continuously. This variation of the bending moment, shear, or any similar function at a given fixed point due to a moving load can be represented graphically by a curve (or straight line) called an **influence line.**

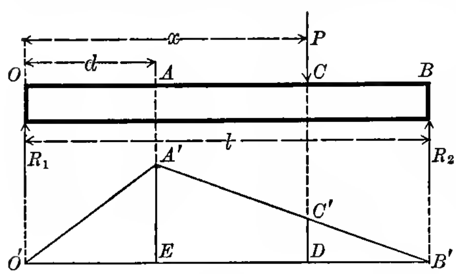


Fig. 77

To obtain the influence line for bending moment for a simple beam of length l, let d denote the distance of the given point A from the left support O, and x the distance of a movable load P from O (Fig. 77). Then, if P is on the right of A, $R_1 = \frac{P(l - x)}{l}$, and hence the moment at A is

$$M_a = \frac{P(l - x)d}{l}.$$

Now let P be a unit load (say one pound or one ton). Then

$$M_a = \frac{(l - x)d}{l};$$

* For a brief course the remainder of this chapter may be omitted.

and if the values of M_a corresponding to each value of x from d to l are laid off as ordinates, we obtain the straight line $A'B'$, which therefore represents the variation in the bending moment at the point A as the unit load moves from B to A. Similarly, if the unit load is on the left of A, $M_a = \frac{x(l-d)}{l}$, which is the equation of the straight line $O'A'$. At D' both lines have the same ordinate, namely, $A'E = \frac{d(l-d)}{l}$. The influence line for bending moment is therefore the broken line $O'A'B'$.

From this construction, it is obvious that the ordinate to the influence line at any point D represents the bending moment at A due to a unit load at D. Thus, as a unit load comes on the beam from the right, the bending moment at A increases from the value zero for the load at B to the value $A'E$ for the load at A, and then decreases again to the value zero at O. Therefore, having constructed for a unit load the influence line corresponding to any given point A, the moment at A due to a load P is found by multiplying P by the ordinate to the influence line directly under P.

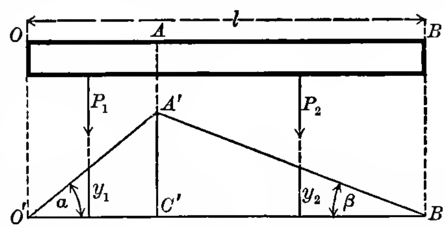


FIG. 78

Problem 139. Find the position of a system of moving loads on a beam so that the bending moment at any point A shall be a maximum.

Solution. Let $O'A'B'$ be the influence line for bending moment for the point A, and let the loads on each side of A be replaced by their resultants P_1 and P_2 (Fig. 78). Then, if y_1 and y_2 are the ordinates to the influence line directly under P_1 and P_2, the moment at A is

$$M_a = P_1 y_1 + P_2 y_2.$$

Now, if the loads move a small distance dx to the left, the moment at A becomes

$$M_a + dM_a = P_1(y_1 - dx \tan\alpha) + P_2(y_2 + dx \tan\beta).$$

Therefore, by subtraction,

$$dM_a = -P_1\,dx \tan\alpha + P_2\,dx \tan\beta,$$

and hence

$$\frac{dM_a}{dx} = -P_1 \tan\alpha + P_2 \tan\beta.$$

For a maximum value of M_a, $\frac{dM_a}{dx} = 0$, in which case

$$P_2 \tan\beta = P_1 \tan\alpha.$$

This equation may be written

$$P_2 \frac{A'C'}{C'B'} = P_1 \frac{A'C'}{O'C'},$$

or

$$\frac{P_2}{C'B'} = \frac{P_1}{O'C'},$$

from which, by composition,

$$\frac{P_1 + P_2}{O'B'} = \frac{P_1}{O'C'},$$

which is the criterion for maximum moment at A. Expressed in words, *the moment at any point A is a maximum when the unit load on the whole span is equal to the unit load on the smaller segment.*

76. Influence line for shear. To obtain the influence line for shear, let l, d, and x have the same meaning as in the preceding article. The shear at any point A is equal to the reaction at O, and for a unit load this reaction is

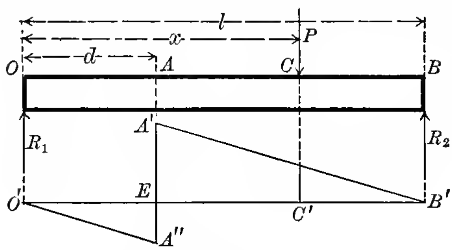


FIG. 79

$$R_1 = \frac{l - x}{l}.$$

If, then, the values of R_1 for all values of x from d to l are laid off as ordinates, the locus of their ends will be the straight line $B'A'$ (Fig. 79). Similarly, for a unit load on the left of A the shear at A is negative, and its amount is $-R_2 = -\frac{x}{l}$, which is the equation of the straight line $O'A''$. Since the slopes of the two lines $A'B'$ and $O'A''$ are equal, these lines are parallel. The influence line for shear is, then, the broken line $O'A''A'B'$.

As a load comes on the beam from the right the shear at A gradually increases from the value zero for the load at B to the value $A'E$ for the load just to the right of A. As the load passes A the shear at this point suddenly decreases by the amount of the load, thus becoming negative, and then increases until the load reaches O, when it again becomes zero. Consequently, the shear at A, due to a load P at

any point C, is found by multiplying P by the ordinate to the influence line at C', directly under C.

Problem 140. Find the position of a system of moving loads on a beam so that the shear at any point A shall be a maximum.

Solution. Let the influence line for the point A be as represented in Fig. 80. Also let P_1 and P_2 be two consecutive loads, d the distance between them, and P' the resultant of all the loads on the beam. Since $A'E$ is the maximum ordinate to the influence line, the maximum shear at A must occur when one of the loads is just to the right of A. Suppose the load P_1 is just to the right of A. Then as P_1 passes A the shear at A is suddenly decreased by the amount P_1. If the loads continue to move to the left until P_2 reaches A, the shear is gradually increased by the amount $P'd \tan \alpha$, since the ordinate under each load is increased by the amount $d \tan \alpha$. Consequently, either P_1 or P_2 at A will give the maximum shear at this point according as

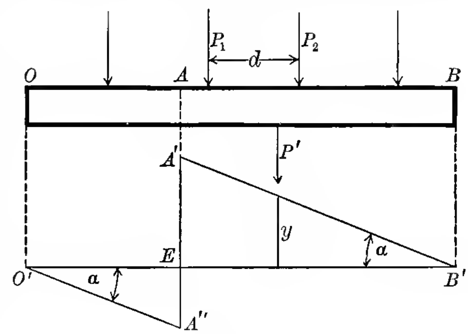


FIG. 80

$$P_1 \gtrless P'd \tan \alpha ;$$

or, since $\tan \alpha = \frac{1}{l}$, according as

$$\frac{P_1}{d} \gtrless \frac{P'}{l}.$$

By means of this criterion, it can be determined in any given case which of two consecutive loads will give the greater shear at any point.

77. Maxwell's theorem. When a load is brought on a beam it causes every point of the beam to deflect, the amount of this deflection for any point being the corresponding ordinate to the elastic curve. If, then, a number of loads rest on a beam, the deflection at any point of the beam is the sum of the deflections at this point due to each of the loads taken separately.

For example, if two loads P_1 and P_2 rest on a beam at the points A and B respectively, the deflection at one of these points, say A, is composed of two parts, namely, the deflection at A due to P_1 and the deflection at A due to P_2. Similarly, the total deflection at B is composed of the partial deflections due to P_1 and P_2 respectively.

Maxwell's theorem, when modified so as to apply to beams, states that *if unit loads rest on a beam at two points I and K, the deflection at I due to the unit load at K is equal to the deflection at K due to the unit load at I.* The following simple proof of the theorem is due to Föppl.*

Consider a simple beam bearing unit loads at two points I and K (Fig. 81). Let the deflection at K due to a unit load at I be denoted by J_{ki}, the deflection at I due to a unit load at I by J_{ii}, etc., the second subscript in each case denoting the point at which the unit load is applied, and the first subscript the point for which the number gives the deflection. Thus J_{ik} denotes the influence of a unit load at K on the deflection at I. For this reason the quantity J_{ik} is called an **influence number**.

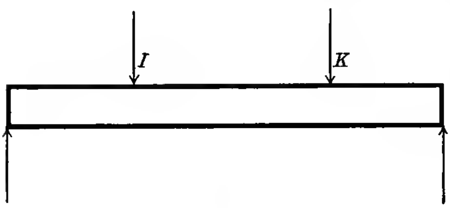


FIG. 81

If the load at I is of amount P_i, the deflection at I is $J_{ii}P_i$, that at K is $J_{ki}P_i$, etc.

Now suppose that a load P_i is brought on the beam gradually at the point I. Then its average value is $\frac{1}{2}P_i$, the deflection under the load is $J_{ii}P_i$, and consequently the work of deformation is $\frac{1}{2}P_i(J_{ii}P_i)$. After the load P_i attains its full value suppose that a load P_k is brought on gradually at K. Then the average value of this load is $\frac{1}{2}P_k$, but since P_i keeps its full value during this second deflection, the work of deformation in this movement is $P_i(J_{ik}P_k) + \frac{1}{2}P_k(J_{kk}P_k)$. Therefore the total work of deformation from both deflections is

$$W = \tfrac{1}{2}J_{ii}P_i^2 + J_{ik}P_iP_k + \tfrac{1}{2}J_{kk}P_k^2.$$

Evidently the same amount of work would have been done if the load P_k had first been applied, and then P_i. The expression for the total work obtained by applying the loads in this order is

$$W = \tfrac{1}{2}J_{kk}P_k^2 + J_{ki}P_hP_k + \tfrac{1}{2}J_{ii}P_i^2.$$

Therefore, equating the two expressions for the work of deformation,

$$J_{ik} = J_{ki},$$

which proves the theorem.

* *Festigkeitslehre*, p. 197.

Problem 141. A beam bears a load of 15 tons at a certain point A, and its deflections at three other points, B, C, D, are measured and found to be .30 in., .15 in., and .09 in. respectively. If loads of 5, 12, and 8 tons are brought on at B, C, and D respectively, find the deflection at A.

Solution. The deflections at B, C, and D due to a *unit* load (one ton) at A are $\frac{.30}{15} = .02$ in., $\frac{.15}{15} = .01$ in., and $\frac{.09}{15} = .006$ in. respectively. Therefore, by Maxwell's theorem, the deflection at A is

$$D_a = .02 \times 5 + .01 \times 12 + .006 \times 8 = .268 \text{ in.}$$

78. Influence line for reactions. The most important application of Maxwell's theorem is to the determination of the unknown reactions for a continuous beam.

Consider a beam continuous over three supports, as shown in Fig. 82. Suppose the middle support removed and a unit load (say 1 ton) placed at this point. Then, if the elastic curve is plotted, the ordinate to this curve at any point I is the deflection at I due to the unit load at B, or, in other words, this ordinate is the influence number J_{ib}. Similarly, the ordinate to the elastic curve at B is the influence number J_{bb}.

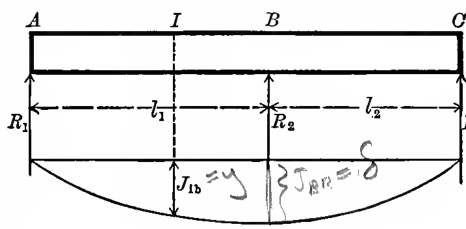


FIG. 82

Now R_2, the unknown reaction at B, must be of such amount as to counteract the deflection at B due to a load P at any point I. Therefore

$$R_2 J_{bb} = P J_{bi}.$$

But, by Maxwell's theorem, $J_{bi} = J_{ib}$; consequently

$$R_2 = P\left(\frac{J_{ib}}{J_{bb}}\right).$$

The influence numbers J_{ib} and J_{bb} are known as soon as the elastic curve for unit load at B is plotted. Therefore, in this case, the construction of one elastic curve gives sufficient data for all further calculations.

Since for any point I the fraction $\frac{J_{ib}}{J_{bb}}$ is proportional to J_{ib} (the denominator being constant), the elastic curve is called the **influence line for reactions.**

For a number of concentrated loads $P_1, P_2, \cdots, P_n$ the same method applies, R_2 in this case being given by the equation

$$R_2 = \frac{J_{1b}}{J_{bb}} P_1 + \frac{J_{2b}}{J_{bb}} P_2 + \cdots + \frac{J_{nb}}{J_{bb}} P_n,$$

or, more briefly,

$$R_2 = \frac{1}{J_{bb}} \sum_i^{1 \cdots n} J_{ib} P_i.$$

To determine the reactions for a beam continuous over four supports and bearing a single concentrated load P at any point I, suppose the two middle supports removed. Then if a unit load is placed at B (Fig. 83) and the elastic curve drawn, the ordinate to this curve at any point I is the influence number J_{ib}. Similarly, by placing a unit load at C and constructing the corresponding elastic curve, the influence number J_{ic} is obtained. Now the reaction R_2 must be of such amount as to counteract the deflections at B due to a load P at I and a load R_3 at C. Therefore

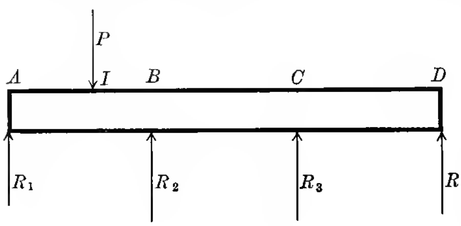


FIG. 83

$$R_2 J_{bb} = P J_{bi} - R_3 J_{bc}.$$

Similarly the reaction R_3 must be of such amount as to counteract the deflections at C due to a load P at I and a load R_2 at B. Therefore

$$R_3 J_{cc} = P J_{ci} - R_2 J_{cb}.$$

By Maxwell's theorem, $J_{bi} = J_{ib}$ and $J_{ci} = J_{ic}$. Making these substitutions and solving the above equations simultaneously for R_2 and R_3,

$$R_2 = P \frac{J_{ib} J_{cc} - J_{ic} J_{bc}}{J_{bb} J_{cc} - J_{bc}^2},$$

$$R_3 = P \frac{J_{ic} J_{bb} - J_{ib} J_{bc}}{J_{bb} J_{cc} - J_{bc}^2}.$$

79. Castigliano's theorem. Consider a beam bearing any number of concentrated loads $P_1, P_2, \cdots, P_n$, acting either vertically upward or downward, and let W denote the work of deformation due to these loads (Fig. 84). Then if one of the loads, say P_i, is increased by a small amount dP_i, the deflection of P_1 is increased by the amount $J_{1i}dP_i$, that of P_2 by the amount $J_{2i}dP_i$, etc., where J_{1i}, J_{2i}, etc., are influence numbers. Therefore the work of deformation is increased by the amount

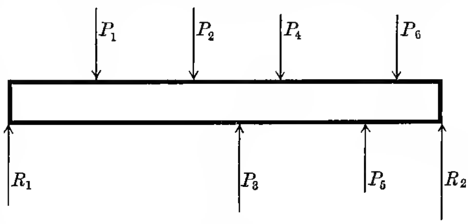


FIG. 84

$$dW = P_1J_{1i}dP_i + P_2J_{2i}dP_i + \cdots + P_nJ_{ni}dP_i;$$

whence

$$\frac{dW}{dP_i} = P_1J_{1i} + P_2J_{2i} + \cdots + P_nJ_{ni}.$$

In forming this expression the work done by dP_i itself has been neglected, since it is infinitesimal in comparison with that done by P_1, P_2, etc.

Now, from Maxwell's theorem, $J_{ik} = J_{ki}$. Therefore the above expression becomes

$$\frac{dW}{dP_i} = P_1J_{i1} + P_2J_{i2} + \cdots + P_nJ_{in}.$$

The right member of this equality, however, is the total deflection D_i at the point I, due to all the loads. Consequently the above expression may be written

$$\frac{dW}{dP_i} = D_i.$$

Since the work of deformation W is a function of all the loads and not of P_i only, this latter expression should be written as a partial derivative; thus

$$\frac{\partial W}{\partial P_i} = D_i,$$

and in this form it is the algebraic statement of Castigliano's theorem. Expressed in words, the theorem is: *The deflection of the point of application of an external force acting on a beam is equal to the partial derivative of the work of deformation with respect to this force.*

80. Application of Castigliano's theorem to continuous beams. Castigliano's theorem affords still another means of determining the unknown reactions of a continuous beam; for the reactions may be included among the loads on the beam, and since the points of application of these reactions are assumed to be fixed, their deflections are zero. Therefore, if P_k is one of the reactions, $D_k = 0$, and consequently

$$\frac{\partial W}{\partial P_k} = 0.$$

A condition equation of this kind can be found for each reaction, and from the system of simultaneous equations so obtained the unknown reactions may be calculated. The following problems illustrate the application of the theorem.

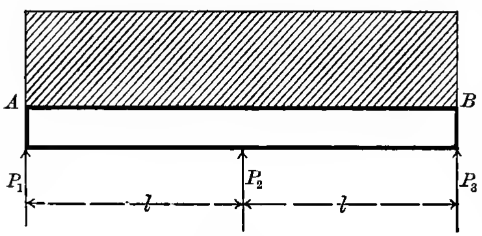


FIG. 85

Problem 142. A uniformly loaded beam of length $2l$ is supported at its center and ends. Find the reactions of the supports by means of Castigliano's theorem.

Solution. Let w denote the unit load on the beam (Fig. 85). From symmetry, $P_1 = P_3$. Also, by taking moments about B,

$$P_1 = wl - \frac{P_2}{2} = P_3.$$

For a point in the first opening at a distance x from the left support,

$$M = P_1 x - \frac{wx^2}{2};$$

consequently,

$$W = \frac{1}{2}\int_0^l \frac{M^2 dx}{EI} = \frac{1}{2\,EI}\left[\frac{P_1^2 l^3}{3} - \frac{P_1 w l^4}{4} + \frac{w^2 l^5}{20}\right].$$

The work of deformation for the other half of the beam is of the same amount. Therefore the total work of deformation is

$$W = \frac{1}{EI}\left[\frac{P_1^2 l^3}{3} - \frac{P_1 w l^4}{4} + \frac{w^2 l^5}{20}\right].$$

Since P_1 is a function of P_2, the partial derivative of W with respect to P_2 is

$$\frac{\partial W}{\partial P_2} = \frac{1}{EI}\left[\frac{2\,P_1 l^3}{3}\cdot\frac{\partial P_1}{\partial P_2} - \frac{wl^4}{4}\cdot\frac{\partial P_1}{\partial P_2}\right].$$

Since $P_1 = wl - \dfrac{P_2}{2}$, $\dfrac{\partial P_1}{\partial P_2} = -\dfrac{1}{2}$.

Therefore
$$\frac{\partial W}{\partial P_2} = \frac{l^3}{EI}\left[\frac{wl}{8} - \frac{P_1}{3}\right].$$

By Castigliano's theorem, $\frac{\partial W}{\partial P_2} = 0$. Therefore
$$\frac{l^3}{EI}\left[\frac{wl}{8} - \frac{P_1}{3}\right] = 0;$$
whence
$$P_1 = \tfrac{3}{8} wl.$$

Substituting in this expression the value of P_1 in terms of P_2,
$$P_2 = \tfrac{5}{4} wl.$$

Problem 143. A uniformly loaded beam extends over three openings of equal span. Find the reactions at the supports.

Solution. Let l denote the length of each span and w the unit load (Fig. 86).

From symmetry, $P_1 = P_4$ and $P_2 = P_3$. Also, by taking moments about B,
$$P_1 = \frac{3\,wl}{2} - P_2 = P_4.$$

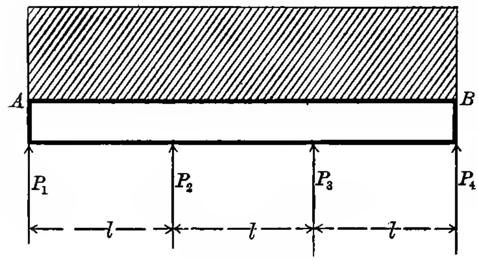


FIG. 86

For any point in the first opening at a distance x from the left support,
$$M = P_1 x - \frac{wx^2}{2},$$
and therefore, as in the preceding problem,
$$W_1 = \frac{1}{2\,EI}\left[\frac{P_1^2 l^3}{3} - \frac{P_1 wl^4}{4} + \frac{w^2 l^5}{20}\right].$$

Since $P_1 = P_4$, W has the same value for the third opening, that is, $W_3 = W_1$. In the second opening
$$M = P_1 x + P_2 (x - l) - \frac{wx^2}{2},$$
and therefore
$$W_2 = \frac{1}{2\,EI}\int_l^{2l}\left(P_1 x + P_2 (x - l) - \frac{wx^2}{2}\right)^2 dx$$
$$= \frac{1}{2\,EI}\left\{\frac{7\,P_1^2 l^3}{3} + \frac{5\,P_1 P_2 l^3}{3} + \frac{31\,w^2 l^5}{20} + \frac{P_2^2 l^3}{3} - \frac{15\,P_1 wl^4}{4} - \frac{17\,P_2 wl^4}{12}\right\}.$$

Hence the total work of deformation for all three openings is
$$W = \frac{1}{2\,EI}\left\{3\,P_1^2 l^3 - \frac{17\,P_1 wl^4}{4} + \frac{33\,w^2 l^5}{20} + \frac{5\,P_1 P_2 l^3}{3} + \frac{P_2^2 l^3}{3} - \frac{17\,P_2 wl^4}{12}\right\}.$$

Therefore
$$\frac{\partial W}{\partial P_2} = \frac{1}{2\,EI}\left\{6\,P_1 l^3 \frac{\partial P_1}{\partial P_2} - \frac{17\,wl^4}{4}\frac{\partial P_1}{\partial P_2} + \frac{5\,P_1 l^3}{3} + \frac{5\,P_2 l^3}{3}\frac{\partial P_1}{\partial P_2} + \frac{2\,P_2 l^3}{3} - \frac{17\,wl^4}{12}\right\}.$$

Since $P_1 = \frac{3\,wl}{2} - P_2$, $\frac{\partial P_1}{\partial P_2} = -1$, and hence

$$\frac{\partial W}{\partial P_2} = \frac{1}{2\,EI}\left\{-\frac{13\,P_1 l^3}{3} + \frac{17\,wl^4}{6} - P_2 l^3\right\}.$$

Putting $\frac{\partial W}{\partial P_2} = 0$, and substituting for P_1 its value in terms of P_2,

$$\frac{-13\,l^3}{3}\left(\frac{3\,wl}{2} - P_2\right) + \frac{17\,wl^4}{6} - P_2 l^3 = 0;$$

whence $$P_2 = \tfrac{11}{10}\,wl,$$

and consequently $$P_1 = \tfrac{2}{5}\,wl.$$

81. Principle of least work. Differentiating partially with respect to P_i both members of the equation $\frac{\partial W}{\partial P_i} = D_i$, we have

$$\frac{\partial^2 W}{\partial P_i^2} = \frac{\partial D_i}{\partial P_i}.$$

As the load increases the deflection increases, and vice versa. Therefore, since ∂D_i and ∂P_i have the same sign, $\frac{\partial D_i}{\partial P_i}$ is positive and hence $\frac{\partial^2 W}{\partial P_i^2}$ is also positive. But, from the differential calculus,

$$\frac{\partial W}{\partial P_i} = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial P_i^2} > 0$$

are the conditions that W shall be a minimum. Consequently, the reactions of a continuous beam, calculated from the condition $\frac{\partial W}{\partial P_i} = 0$, are such that they make the work of deformation a minimum.

In Article 73 it was pointed out that the internal work of deformation is a form of potential energy. The above is thus a special case of what is known as the **principle of least work,** the general statement of this principle being expressed by the following theorem:

For stable equilibrium the potential energy of any system is a minimum.

The importance of the principle of least work is due to the fact that it is a general mechanical principle, affording a general solution of all problems involving the static equilibrium of elastic solids. Its most useful application, perhaps, is to problems which are otherwise statically indeterminate, that is to say, problems in which the

number of unknown quantities involved is greater than the number of relations furnished by the ordinary conditions of equilibrium.

The general solution of any problem of this nature by the method of least work is as follows: First express the work of deformation (or potential energy) in terms of the unknown quantities which it is required to determine. Then the condition that this expression shall be a minimum resolves itself into the condition that the partial derivatives of the potential energy with respect to each of the unknowns involved shall be zero. In this way we obtain exactly as many equations as unknowns, from which these unknown quantities may be found.

Thus if W denotes the work of deformation and $P_1, P_2, \cdots, P_n$ the unknown quantities to be found, first express W as a function of these unknowns, say $W(P_1, P_2, \cdots, P_n)$. Then the condition for a minimum is $dW = 0$, or, expressing the total differential dW in terms of its partial derivatives with respect to the various unknowns,

$$dW = \frac{\partial W}{\partial P_1} dP_1 + \frac{\partial W}{\partial P_2} dP_2 + \cdots + \frac{\partial W}{\partial P_n} dP_n = 0.$$

Since $P_1, P_2, \cdots, P_n$ are assumed to be independent, in order for this relation to be satisfied identically, that is for all values of $P_1, P_2, \cdots, P_n$, the coefficients of $dP_1, dP_2, \cdots, dP_n$ must all be zero; that is,

$$\frac{\partial W}{\partial P_1} = 0, \quad \frac{\partial W}{\partial P_2} = 0, \quad \cdots, \quad \frac{\partial W}{\partial P_n} = 0.$$

We have, therefore, n equations from which to determine the n unknowns $P_1, P_2, \cdots, P_n$.

Before applying this principle it is necessary to find an expression for the work of deformation of elastic solids subjected to direct stress or to bending stress.

1. **Direct stress.** Consider a prismatic bar of length l and cross section F, which is subjected to a direct stress, either tension or compression, of intensity p. Then from Hooke's law

$$\frac{p}{s} = E;$$

or if P denotes the total load, then since $p = \frac{P}{F}$ and $s = \frac{\Delta l}{l}$, this becomes $\frac{Pl}{\Delta l F} = E$, whence $\Delta l = \frac{Pl}{FE}$. If, then, the load is applied gradually, the average force acting on the bar during deformation is $\frac{1}{2}P$, and consequently the work of deformation in this case is

$$W = \frac{P^2 l}{2\,FE}.$$

2. **Bending stress.** The work of deformation of a prismatic beam subjected to a bending moment M has been found in Article 72 to be

$$W = \int_0^l \frac{M^2 dx}{2\,EI}.$$

The application of the method of least work will now be illustrated by a number of simple problems. Problems 142 and 143, Article 80, and Articles 82 and 83 are also applications of this principle.

Problem 144. Three Carnegie I-beams, No. B 80, are placed 4 ft. apart across an opening 25 ft. wide. Across their centers is placed another I-beam of the same dimensions as the first, and upon the center of this cross beam there rests a load of 10 tons. Find the greatest stress which occurs in any member of the construction.

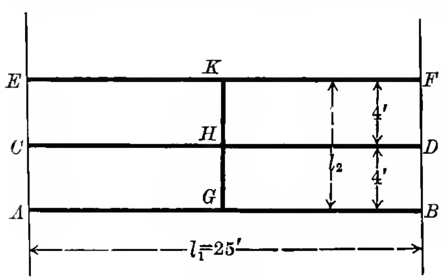


Fig. 87

Solution. Let the amount of the load at H, which is carried by GK, be denoted by P (Fig. 87). Then the loads on AB and EF at G and K respectively are each equal to $\frac{P}{2}$, and the load on CD is 20,000 lb. $- P$.

Now the work of deformation for a simple beam of length l bearing a single concentrated load P' at its center is, from Problem 135,

$$W = \frac{P'^2 l^3}{96\,EI}.$$

Therefore, since the load on AB or EF is $\frac{P}{2}$, the work of deformation for either of these beams is

$$W_{ab} = W_{ef} = \frac{P^2 l_1^3}{384\,EI}.$$

Similarly the work of deformation for CD is

$$W_{cd} = \frac{(20{,}000 - P)^2 l_1^3}{96\,EI},$$

and for GK is

$$W_{gk} = \frac{P^2 l_2^3}{96\,EI}.$$

Hence the total work of deformation for the entire construction is

$$W = \frac{P^2 l_1^3}{192\,EI} + \frac{(20{,}000 - P)^2 l_1^3}{96\,EI} + \frac{P^2 l_2^3}{96\,EI}.$$

By the principle of least work, $\dfrac{\partial W}{\partial P} = 0$; consequently

$$\frac{\partial W}{\partial P} = \frac{P l_1^3}{96\,EI} - \frac{(20{,}000 - P) l_1^3}{48\,EI} + \frac{P l_2^3}{48\,EI} = 0.$$

From the Carnegie handbook, $I = 795.6$ in.^{4}, and from the figure, $l_1 = 300$ in., $l_2 = 96$ in. Inserting these numerical values in the above expression, and solving for P,

$$P = 13{,}048 \text{ lb.}$$

Having determined P, the stress in the various members can easily be calculated. Thus it is found that the greatest stress occurs in CD, its amount being $p = 23{,}593$ lb./in.^{2}

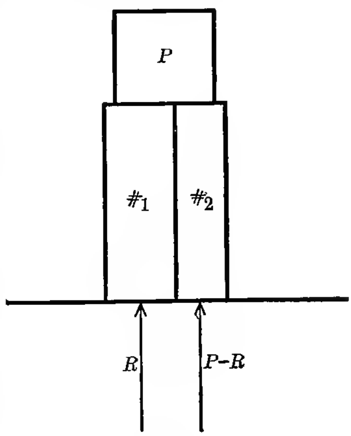


FIG. 88

Problem 145. Two short posts of the same length l but of cross-section areas F_1 and F_2 and of material having moduli E_1 and E_2 carry a load P jointly. How much of the load is carried by each? (Fig. 88.)

Solution. Let R denote the load carried by No. 1. Then the load carried by No. 2 is $P - R$. Hence, applying the expression for the work due to a direct stress, the total work of deformation for both posts is

$$W = \frac{R^2 l}{2\,F_1E_1} + \frac{(P - R)^2 l}{2\,F_2E_2}.$$

The condition for a minimum gives

$$\frac{\partial W}{\partial R} = 0 = \frac{Rl}{F_1E_1} - \frac{(P - R)\,l}{F_2E_2};$$

whence

$$R = \frac{F_1E_1}{F_1E_1 + F_2E_2} \cdot P.$$

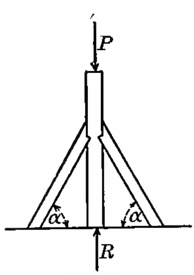


FIG. 89

Problem 146. A post supporting a load P is braced near the bottom by two braces each of length l and inclined at the same angle α to the horizontal (Fig. 89). If the

upright is of cross section F_1 and has a modulus E_1, and the braces are each of cross section F_2 and modulus E_2, show that the load R carried by the upright is given by

$$R = \frac{F_1 E_1 P}{4 \sin^3 \alpha F_2 E_2 + F_1 E_1}.$$

Problem 147. A platform 12 ft. × 18 ft. in size and weighing 1 ton is supported at the corners by four wooden legs, each 8 in. square. A load of 5 tons is placed on this platform 4 ft. from each of two adjacent edges. How much of the load is carried by each leg?

Problem 148. A beam 20 ft. long is supported at each end and at a point distant 5 ft. from the left end. It carries a load of 180 lb. at the left end, and of 125 lb. at a point distant 6 ft. from the right end. Find the reactions of the supports.

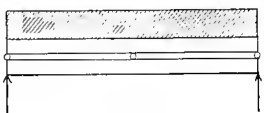


FIG. 90

Problem 149. Two beams are supported as shown in Fig. 90, the lower beam resting on fixed end supports, and the upper beam resting on three supports, at its center and ends. The upper beam carries a uniform load. Find the center load transmitted to the lower beam.

Problem 150. A flitched (or composite) beam consists of a 3-in. I-beam weighing $7\frac{1}{2}$ lb./ft. and a 4-in. × 6-in. timber, the I-beam being placed underneath the wooden beam, and the two are hung from a crane by a wrought-iron strap around the middle. A cable is then looped over the ends of this flitched beam $2\frac{1}{2}$ ft. distant from the center on each side, and a load of 1000 lb. supported by the loop. Find the total load carried by each beam.

Problem 151. The king post truss shown in Fig. 91 is formed of a single beam AC resting on supports at A and C and trussed at the center with a strut BD, supported by two tie rods AD and DC. Determine the load R carried by the strut BD when a load P is placed at a distance c from A.

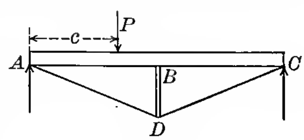


FIG. 91

Solution. Let R denote the stress in BD. Then if h denotes the length of the strut BD and d the length of each tie, AD and DC, the stress in AD or DC is $\frac{Rd}{2h}$ in each, and the direct stress in ABC is $\frac{Rl}{4h}$. Let F_1, F_2, F_3, denote the cross-section areas of AC, AD, and BD respectively. Then the total work of deformation, due to direct stresses in the various members, is

$$W = \frac{R^2 h}{2 F_3 E_3} + \frac{R^2 d^3}{4 h^2 F_2 E_2} + \frac{R^2 l^3}{32 F_1 E_1 h^2}.$$

In addition to this it is also necessary to consider the work of deformation due to the bending stress in AC. At a point distant x from A this is as follows:

For x between A and P, $\quad M_{AP} = \left[\frac{P(l-c)}{l} - \frac{R}{2}\right]x,$

for x between P and B, $\quad M_{PB} = Pc - \left(\frac{R}{2} + \frac{Pc}{l}\right)x,$

for x between B and C, $\quad M_{BC} = \left(\frac{Pc}{l} - \frac{R}{2}\right)(l - x).$

Hence the total internal work due to bending is

$$W=\frac{1}{2\,EI}\left\{\int_0^c\left[\frac{P(l-c)}{l}-\frac{R}{2}\right]^2x^2dx+\int_c^{\frac{l}{2}}\left[Pc-\left(\frac{R}{2}+\frac{Pc}{l}\right)x\right]^2dx\right.$$
$$\left.+\int_{\frac{l}{2}}^{l}\left[\left(\frac{Pc}{l}-\frac{R}{2}\right)(l-x)\right]^2dx\right\}.$$

Now applying the condition $\frac{\partial W}{\partial R}=0$ to the sum of these expressions, and solving the resulting equation for R, we have finally

$$R=\frac{\dfrac{3\,cl^2-4\,c^3}{48\,E_1I_1}}{\dfrac{h}{E_3F_3}+\dfrac{d^3}{2\,h^2E_2F_2}+\dfrac{l^3}{16\,h^2E_1F_1}+\dfrac{l^3}{48\,E_1I_1}}\cdot P.$$

Problem 152. A wooden beam 12 in. deep, 10 in. wide, and 20 ft. long between supports is reënforced by a steel rod 2 in. in diameter and a cast-iron strut 3 in. square and 2 ft. high, the whole forming a king post truss. Find the stress in each member due to a uniform load of 1200 lb./ft. over the entire beam.

82. General formula for flexural deflection. The ordinary method of determining flexural deflection is by computing the ordinate to the elastic curve at the required point, each case requiring separate treatment. A general formula for flexural deflection, however, may be obtained by applying the method of least work in the form of Castigliano's theorem.

From Article 73 the work of deformation due to bending is given by

$$W_B=\int\frac{M^2dx}{2\,EI}.$$

Now in order to apply Castigliano's theorem to this expression, assume a concentrated load K applied to the beam at the point whose deflection is desired, and let this load be subsequently reduced to zero. Let

M = moment at any section due to given loading,
M' = moment at any section due to a *unit* load at a given point.

Then for a load K at the given point, the moment at any section due to this load becomes KM', and hence the total moment due to the given loading and the assumed load K is

$$M_1=M+KM'.$$

Therefore the above expression for the work of deformation now becomes

$$W_B = \int \frac{(M + KM')^2 dx}{2\,EI}.$$

By Castigliano's theorem the actual deflection D_B due to the given loading only is

$$D_B = \left[\frac{\partial W_B}{\partial K}\right]_{K=0},$$

and hence applying this to the expression for W_B, we have

$$D_B = \left[\int \frac{2\,(M + KM')\,\frac{\partial}{\partial K}(M + KM')}{2\,EI}\,dx\right]_{K=0},$$

or, simplifying,

$$D_B = \int \frac{MM'}{EI}\,dx,$$

which is the required general formula for flexural deflection.* All the ordinary formulas for the flexural deflection of beams under various loadings and with different methods of support are simply special cases of this general formula, as illustrated by the following examples.

Problem 153. Find the flexural deflection at the center of a simple beam of constant cross section and bearing a single concentrated load P at the center.

Solution. Here $M = \frac{Px}{2}$, and applying a unit load at the point whose deflection is desired, namely the center, $M' = \frac{x}{2}$. Consequently,

$$D_B = 2\int_0^{\frac{l}{2}} \frac{Px^2dx}{4\,EI} = \frac{Pl^3}{48\,EI}.$$

Problem 154. Find the flexural deflection at the center of a simple beam of constant cross section bearing a uniform load over the entire span.

Solution. In this case $M = \frac{wlx}{2} - \frac{wx^2}{2}$ and $M' = \frac{x}{2}$. Consequently,

$$D_B = 2\int_0^{\frac{l}{2}} \left(\frac{wlx^2}{4} - \frac{wx^3}{4}\right)\frac{dx}{EI} = \frac{5\,wl^4}{384\,EI}.$$

* This formula is due to Professor Fraenkel, but it is believed that the above proof has never before been given.

Problem 155. Find the flexural deflection at the center of a beam of constant cross section fixed at both ends and bearing a single concentrated load at the center.

Solution. The first step in this problem is to determine the moment at one support. This is determined from the condition that the deflection at the support is zero.

Applying a load of unity at the left support, we have for sections on either side of the center,

$$\text{left of center}\begin{cases} M = M_0 - \dfrac{Px}{2}, \\ M' = x, \end{cases} \qquad \text{right of center}\begin{cases} M = M_0 - \dfrac{Px}{2} + P\left(x - \dfrac{l}{2}\right), \\ M' = x, \end{cases}$$

where M_0 denotes the moment at the left support. Substituting these values in the condition

$$D_B \text{ at support} = 0,$$

we have

$$D_B = \frac{1}{EI}\int_0^{\frac{l}{2}}\left(M_0 - \frac{Px}{2}\right)x\,dx + \frac{1}{EI}\int_{\frac{l}{2}}^{l}\left(M_0 + \frac{Px}{2} - \frac{Pl}{2}\right)x\,dx$$

$$= \frac{1}{EI}\left[\frac{M_0x^2}{2} - \frac{Px^3}{6}\right]_0^{\frac{l}{2}} + \frac{1}{EI}\left[\frac{M_0x^2}{2} + \frac{Px^3}{6} - \frac{Plx^2}{4}\right]_{\frac{l}{2}}^{l} = 0,$$

whence

$$M_0 = \frac{Pl}{8}.$$

Proceeding now to find the center deflection, we have

$$M = M_0 - \frac{Px}{2} = \frac{Pl}{8} - \frac{Px}{2},$$

$$M' = \frac{l}{8} - \frac{x}{2},$$

and, consequently,

$$D_B = \frac{2}{EI}\int_0^{\frac{l}{2}}\left(\frac{Pl^2}{64} - \frac{Plx}{16} - \frac{Plx}{16} + \frac{Px^2}{4}\right)dx = \frac{Pl^3}{192\,EI}.$$

Problem 156. Find the flexural deflection at the center of a beam of constant cross section fixed at both ends and bearing a uniform load over the entire span.

Solution. Let M_0 denote the moment at the left support, and w the load in lb./ft. Then

$$M = M_0 - \frac{wlx}{2} + \frac{wx^2}{2},$$

$$M' = x.$$

To find M_0 apply the condition that the deflection at the support is zero. Then

$$D_B \text{ at support} = \int_0^l \frac{\left(M_0x - \dfrac{wlx^2}{2} + \dfrac{wx^3}{2}\right)dx}{EI}$$

$$= \frac{1}{EI}\left[\frac{M_0x^2}{2} - \frac{wlx^3}{6} + \frac{wx^4}{8}\right]_0^l = 0,$$

whence

$$M_0 = \frac{wl^2}{12}.$$

To find the deflection at the center, we have therefore

$$M = M_0 - \frac{wlx}{2} + \frac{wx^2}{2} = \frac{wl^2}{12} - \frac{wlx}{2} + \frac{wx^2}{2},$$

$$M' = \frac{l}{8} - \frac{x}{2} \text{ (from Problem 155),}$$

and, consequently,

$$D_B \text{ at center} = \frac{2}{EI}\int_0^{\frac{l}{2}}\left(\frac{wl^3}{96} - \frac{wl^2x}{16} + \frac{wlx^2}{16} - \frac{wl^2x}{24} + \frac{wlx^2}{4} - \frac{wx^3}{4}\right)dx = \frac{wl^4}{384\,EI}.$$

83. General formula for shearing deflection. A general formula for shearing deflection of beams may also be obtained by the method of least work. For this purpose let W_S denote the work of deformation due to shear, and G the shear modulus. Then if q_1 denotes the unit shearing stress, Hooke's law for shear reads

$$\phi = \frac{q_1}{G},$$

and the unit work of shearing deformation for an infinitesimal parallelopiped of unit volume becomes

$$dW_S = \frac{1}{2}\,q_1\phi dV = \frac{q_1^2}{2\,G}\,dV.$$

Therefore, since $dV = dFdx$, the total work of shearing deformation for the entire beam is

$$W_S = \int dx \int \frac{q_1^2 dF}{2\,G}.$$

Now to determine the shearing deflection, assume a concentrated load K applied to the beam at the point whose deflection is desired, and having used K as required by Castigliano's theorem, let it be subsequently reduced to zero.

For this purpose let

Q = total shear on any variable section due to the given loading,
Q' = shear on any variable section due to a *unit* load at a given point,
q = unit shearing stress due to total shear Q as above,
q' = unit shearing stress due to shear Q'.

Then for a concentrated load K at any given point the shear on any section is $Q'K$, and the unit shear at a variable point due to this load

is $q'K$. Hence the total unit shear due to both the actual given loading and the assumed concentrated load K becomes

$$q_1 = q + q'K.$$

Hence the expression for the work of shearing deformation now becomes

$$W_S = \int dx \int \frac{(q + q'K)^2 \, dF}{2\,G}.$$

Now by Castigliano's theorem the actual shearing deflection due to the *given* loading is

$$D_S = \left[\frac{\partial W}{\partial K}\right]_{K=0}.$$

Performing the indicated differentiation and substitution, we have therefore

$$D_S = \int dx \left[\int \frac{2\,(q + q'K)\,\dfrac{\partial}{\partial K}(q + q'K)\, dF}{2\,G}\right]_{K=0} = \int dx \int \frac{qq'dF}{G}.$$

To simplify this expression, assume the straight-line law of distribution of stress, namely $\dfrac{q}{Q} = \dfrac{q'}{Q'}$ or $q' = \dfrac{Q'}{Q}q$, whence finally

$$\boldsymbol{D_S = \int \frac{Q'}{QG}\, dx \int q^2 dF,}$$

which is the required general formula for shearing deflection. The method of applying this general formula is illustrated below.*

Special Case I. Beam of constant rectangular cross section of height h.

From equation (**28**), Article 56, the unit shear at any point of a cross section bounded by parallel sides is

$$q = \frac{Q}{bI} \int y dF,$$

and from equation (**29**), for a rectangular cross section of height h this becomes

$$q = \frac{Q}{I}\left(\frac{h^2}{8} - \frac{y^2}{2}\right).$$

* For applications of this method to beams of variable cross section see article by S. E. Slocum, in *Journal Franklin Institute*, April, 1911.

Substituting this value of q in the second integral of the general formula, it becomes for the special case under consideration

$$\int q^2 dF = \frac{Q^2}{I^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{h^2}{8} - \frac{y^2}{2}\right)^2 b\,dy$$

$$= \frac{144\,Q^2}{b^2h^6}\cdot 2\int_0^{\frac{h}{2}}\left(\frac{h^4}{64} - \frac{h^2y^2}{8} + \frac{y^4}{4}\right)b\,dy = \frac{6}{5}\frac{Q^2}{F}.$$

Hence the formula for the shearing deflection reduces in this case to the simple integral

$$D_S = \int_0^l \frac{Q'}{QG}\,dx\left[\frac{6}{5}\frac{Q^2}{F}\right] = \frac{6}{5}\int_0^l \frac{QQ'}{FG}\,dx.$$

Problem 157. Determine the shearing deflection at the center of a simple beam of constant rectangular cross section due to a single concentrated load at the center.

Solution. Let P denote the load at center. Then the total shear on any section is

$$Q = \frac{P}{2}.$$

Also assuming a unit load at the point whose deflection is to be determined, namely the center, we have

$$Q' = \tfrac{1}{2}.$$

Substituting these values in the above formula, the shearing deflection D_S is

$$D_S = \frac{6}{5}\cdot 2\int_0^{\frac{l}{2}} \frac{\frac{P}{4}\,dx}{FG} = \frac{3\,Pl}{10\,Gbh}.$$

To determine the relative amounts of the shearing and flexural deflections, assume the relation between the two moduli G and E as $G = \frac{2}{5}E$. Then

$$\frac{D_S}{D_B} = \frac{\frac{3}{4}\frac{Pl}{Ebh}}{\frac{Pl^3}{4\,Ebh^3}} = 3\left(\frac{h}{l}\right)^2.$$

Hence the relative value of the shearing and bending deflections depends in this case on the square of the ratio of the depth of the beam to its length. Thus if $h = \frac{1}{5}l$, $D_S = .12\,D_B$; if $h = \frac{1}{10}l$, $D_S = .03\,D_B$; if $h = \frac{1}{20}l$, $D_S = .0075\,D_B$, etc. The relative dimensions for which the shearing deflection ceases to be of importance are thus easily determined.

Special Case II. Beam of constant circular cross section of radius r. From Article 57, the expression for the unit shear, namely

$$q = \frac{Q}{bI}\int y\,dF,$$

becomes in the case of a circular cross section

$$q = \frac{Qx^2}{3\,\pi r^4}\cdot$$

Substituting this value of q in the second integral of the general formula, we have

$$\int q^2 dF = \int \frac{Q^2x^4}{(3\,\pi r^4)^2}\,x\,dy = \frac{32\,Q^2}{(3\,\pi r^4)^2}\int_{-r}^{r}\left(\frac{x}{2}\right)^5 dy$$

$$= \frac{32\,Q^2}{(3\,\pi r^4)^2}\int_{-r}^{r}(r^2 - y^2)^{\frac{5}{2}}dy = \frac{64\,Q^2}{(3\,\pi r^4)^2}\int_{0}^{r}(r^2 - y^2)^{\frac{5}{2}}dy$$

$$= \frac{10\,\pi r^6 Q^2}{9\,\pi^2 r^8} = \frac{10}{9}\,\frac{Q^2}{F}\cdot$$

Consequently, the shearing deflection in this case is

$$D_S = \int \frac{Q'}{QG}\,dx\left[\frac{10}{9}\,\frac{Q^2}{F}\right] = \frac{10}{9}\int \frac{QQ'}{FG}\,dx.$$

Problem 158. Determine the shearing deflection at the center of a simple beam of constant circular cross section and bearing a uniform load.

Solution. Let the uniform load be of amount w lb. per unit of length. Then $Q = \frac{wl}{2} - wx$. Also assuming a unit load at the point whose deflection is to be determined, namely the center, we have $Q = \frac{1}{2}$. Hence in the present case the shearing deflection at the center is

$$D_S = \frac{10}{9}\cdot 2\int_0^{\frac{l}{2}} \frac{\left(\frac{wl}{2} - wx\right)\frac{1}{2}}{FG}\,dx = \frac{5\,wl^2}{36\,\pi r^2 G}\cdot$$

The relative amount of the shearing deflection as compared with the flexural deflection is, in this case, given by the ratio

$$\frac{D_S}{D_B} = \frac{\dfrac{5\,wl^2}{36\,\pi r^2 G}}{\dfrac{5\,wl^4}{384\,EI}};$$

or assuming, as above, that $G = \frac{2}{5} E$ and denoting the depth of the beam by $h = 2r$, this becomes

$$\frac{D_S}{D_B} = \frac{5}{3}\left(\frac{h}{l}\right)^2.$$

For a circular cross section, therefore, the shearing deflection is of less relative importance for a given ratio of depth to length than for a rectangular cross section. Thus, if $h = \frac{1}{10} l$, $D_S = .016\, D_B$, etc.

EXERCISES ON CHAPTER IV

Problem 159. In building construction the maximum allowable deflection for plastered ceilings is $\frac{1}{360}$ of the span. A floor is supported on 2 in. $\times$ 10 in. wooden joists, 14 ft. span and spaced 16 in. apart on centers. Find the maximum load per square foot of floor surface in order that the deflection may not exceed the amount specified.

Problem 160. Determine the proper spacing center to center for 12-in. steel I-beams weighing 35 lb./ft. for a span of 20 ft. and a uniform floor load of 100 lb./ft.^{2} in order that the deflection shall not exceed $\frac{1}{360}$ of the span.

Problem 161. One end of a beam is built into a wall and the other end is supported at the same level by a post 12 ft. from the wall. The beam carries a uniform load of 100 lb. per linear foot. Find the position and amount of the maximum moment and also of the maximum deflection.

Problem 162. One end of a beam is built into a wall and the other end rests on a prop 20 ft. from the wall, at the same level. The beam bears a concentrated load of 1 ton at a point 8 ft. from the wall. Find the position and amount of the maximum moment and also of the maximum deflection.

Problem 163. A simple beam of length l carries a distributed load which varies uniformly from 0 at one end to w lb. per unit of length at the other. Find the maximum deflection.

Hint. Note that in the notation of Article 67,

$$EI\frac{d^4y}{dx^4} = \text{load per unit length},$$

$$EI\frac{d^3y}{dx^3} = \text{shear},$$

$$EI\frac{d^2y}{dx^2} = \text{moment},$$

$$EI\frac{dy}{dx} = EI \times \text{slope of elastic curve},$$

$$EIy = EI \times \text{deflection}.$$

In the present case, taking the origin at the light end,

$$EI\frac{d^4y}{dx^4} = \frac{wx}{l},$$

which may be integrated to obtain the deflection.

Problem 164. A beam of uniform strength is one whose moment of resistance is in the same constant ratio to the bending moment throughout, so that the skin stress is constant.

Show that in order for a cantilever bearing a single concentrated load at the end to be of uniform strength, if the depth is constant, the plan of the beam must be triangular; whereas if the breadth is constant, the side elevation of the beam must be parabolic.

Problem 165. A structural steel shaft 8 in. in diameter and 5 ft. long between bearings carries a 25-ton flywheel midway between the bearings. Find the maximum deflection of the shaft, considering it as a simple beam.

Problem 166. A wrought-iron bar 2 in. square is bent to a right angle 4 ft. from one end. The other end is then imbedded in a concrete block so that it stands upright with the 4 ft. length horizontal. If the upright projects 12 ft. above the concrete and a load of 300 lb. is hung at the end of the horizontal arm, find the deflection at the end of this arm.

Problem 167. A cantilever of length l is loaded uniformly. At what point of its length should a prop be placed, supporting the beam at the same level as the fixed end, in order to reduce the bending stress as much as possible, and what proportion of the load is then carried by the prop?

Problem 168. A continuous beam extends over three spans of 20 ft., 40 ft., and 30 ft., and carries uniform loads of 3, 1, and 2 tons per linear foot on the three spans respectively. Find the danger sections and the reactions of the supports.

Problem 169. A carriage spring is $2\frac{1}{2}$ ft. long and is built up of steel leaves each 2 in. wide and $\frac{3}{8}$ in. thick. How many leaves are required to carry a central load of 1000 lb. with a factor of safety of 4, and what is the deflection under this load?

HINT. Consider the material spread out in the form of a triangle of constant depth $\frac{3}{8}$ in. and varying width, fixed at the base and carrying the load at the apex. Also compare with Problem 164.

CHAPTER V

COLUMNS AND STRUTS

84. Nature of compressive stress. When a prismatic piece of length equal to several times its breadth is subjected to axial compression it is called a **column**, or **strut**, the word " column " being used to designate a compression member placed vertically and bearing a static load; all other compression members being called struts.

If the axis of a column or strut is not perfectly straight, or if the load is not applied exactly at the centers of gravity of its ends, a bending moment is produced which tends to make the column deflect sideways, or "buckle." The same is true if the material is not perfectly homogeneous, causing certain parts to yield more than others. Such lateral deflection increases the bending moment, and consequently increases the tendency to buckle. A compression member is, therefore, in a different condition of equilibrium from one subjected to tension, for in the latter any deviation of the axis from a straight line tends to be diminished by the stress instead of increased.

The oldest theory of columns is due to Euler, and his formula is still the standard for comparison. Euler's theory, however, is based upon the assumptions that the column is perfectly straight, the material perfectly homogeneous, and the load exactly centered at the ends, — assumptions which are never exactly realized. For practical purposes, therefore, it has been found necessary to modify Euler's formula in such a way as to bring it into accord with the results of actual experiments, as explained in the following articles.

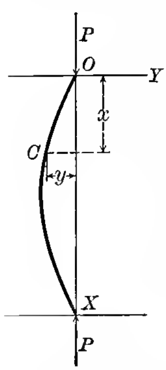


FIG. 92

85. Euler's theory of long columns. Consider a long column subjected to axial loading, and assume that the column is perfectly straight and homogeneous, and that the load is applied exactly at the centers of gravity of its ends.

Assume also that the ends of the column are free to turn about their centers of gravity, as would be the case, for example, in a column with round or pivoted ends.

Now suppose that the column is bent sideways by a lateral force, and let P be the axial load which is just sufficient to cause the column to retain this lateral deflection when the lateral force is removed. Let OX and OY be the axes of X and Y respectively (Fig. 92). Then if y denotes the deflection of a point C at a distance x from O, the moment at C is $M = Py$. Therefore the differential equation of the elastic curve assumed by the center line of the column is

$$EI\frac{d^2y}{dx^2} = -Py,$$

which may be written

$$EI\frac{d^2y}{dx^2} + Py = 0.$$

To integrate this differential equation, multiply by $2\frac{dy}{dx}$.* Then

$$2\frac{d^2y}{dx^2}\frac{dy}{dx} + \frac{2P}{EI}y\frac{dy}{dx} = 0;$$

and integrating each term,

$$\left(\frac{dy}{dx}\right)^2 + \frac{Py^2}{EI} = C_1,$$

where C_1 is a constant of integration. This equation can now be written

$$\frac{dy}{\sqrt{\frac{EIC_1}{P} - y^2}} = \sqrt{\frac{P}{EI}}\,dx.$$

Integrating again,

$$\sin^{-1}\frac{y}{\sqrt{\frac{EIC_1}{P}}} = \sqrt{\frac{P}{EI}}\,x + C_2,$$

where C_2 is also a constant of integration; whence

$$y = \sqrt{\frac{EIC_1}{P}}\sin\left(\sqrt{\frac{P}{EI}}\,x + C_2\right),$$

* This is called an integrating factor and makes each term a perfect differential. See Granville's *Calculus*, pp. 438, 444.

or, expanding,

$$y = \sqrt{\frac{EIC_1}{P}}\left[\sin\left(x\sqrt{\frac{P}{EI}}\right)\cos C_2 + \cos\left(x\sqrt{\frac{P}{EI}}\right)\sin C_2\right].$$

Now for convenience let the constants in this integral be denoted by A, B, and C respectively; that is to say, let

$$A = \sqrt{\frac{EIC_1}{P}}\cos C_2; \quad B = \sqrt{\frac{EIC_1}{P}}\sin C_2; \quad C = \sqrt{\frac{P}{EI}}.$$

Then the general integral becomes

$$y = A \sin Cx + B \cos Cx.$$

At the ends O and X, where $x = 0$ and l, $y = 0$. Substituting these values in the above integral,

$$B = 0, \quad \text{and} \quad A \sin Cl = 0.$$

Since A and B cannot both be zero, $\sin Cl = 0$; whence

$$Cl = \sin^{-1} 0 = \lambda\pi,$$

where λ is an arbitrary integer. Now let λ take the smallest value possible, namely 1, and substitute for C its value. Then

$$l\sqrt{\frac{P}{EI}} = \pi;$$

whence

(48) $$\boldsymbol{P = \frac{\pi^2 EI}{l^2}},$$

which is **Euler's formula for long columns.**

Under the load P given by this formula the column is in neutral equilibrium; that is to say, the load P is just sufficient to cause it to retain any lateral deflection which may be given to it. For this reason P is called the *critical load.* If the load is less than this critical value, the column is in stable equilibrium, and any lateral deflection will disappear when its cause is removed. If the load exceeds this critical value, the column is in unstable equilibrium, and the slightest lateral deflection will rapidly increase until rupture occurs.

86. Columns with one or both ends fixed. The above deduction of Euler's formula is based on the assumption that the ends of the

column are free to turn, and therefore formula (**48**) applies only to long columns with round or pivoted ends.

If the ends of a column are rigidly fixed against turning, the elastic curve has two points of inflection, say B and D. From symmetry, the tangent to the elastic curve at the center C must be parallel to the original position of the axis of the column AE, and therefore the portion AB of the elastic curve must be symmetrical with BC, and CD with DE. Consequently, the points of inflection, B and D, occur at one fourth the length of the column from either end. The critical load for a column with fixed ends is, therefore, the same as for a column with free ends of half the length; whence, *for fixed ends*, Euler's formula becomes

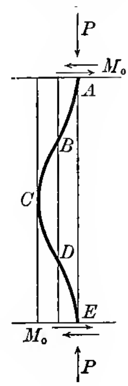


FIG. 93

$$(49) \qquad P = \frac{4\pi^2 EI}{l^2}.$$

Columns with flat ends, fixed against lateral movement, are usually regarded as coming under formula (**49**), the terms "fixed ends" and "flat ends" being used interchangeably.

If one end of the column is fixed and the other end is free to turn, the elastic curve is approximately represented by the line $BCDE$ in Fig. 93. Therefore the critical load in this case is approximately the same as for a column with both ends free, of length BCD, that is, of length equal to $\frac{2}{3}$ BE or $\frac{2}{3}\, l$; whence, for a column with *one end fixed and the other free*, Euler's formula becomes

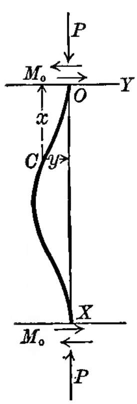


FIG. 94

$$(50) \qquad P = \frac{9\pi^2 EI}{4\,l^2}, \text{ approximately.}$$

87. Independent proof of formulas for fixed ends. The results of the preceding article can be established independently as follows.

Suppose both ends of the column fixed against turning by a moment M_0 at each support. Then the moment at any point C, distant x from 0 (Fig. 94), is $M = -M_0 + Py$, and therefore the equation of the elastic curve is

$$EI\frac{d^2y}{dx^2} = M_0 - Py.$$

Proceeding as in Article 85, the general integral of this equation is found to be

$$y = A\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P},$$

in which A and B are undetermined constants. For $x = 0$ and l, $y = 0$, and $\frac{dy}{dx} = 0$. Therefore, by substituting these values in the general integral, the following relations are obtained:

$$A = 0, \qquad B = -\frac{M_0}{P}, \qquad B\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} = 0,$$

$$-B\sqrt{\frac{P}{EI}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0.$$

From these conditions,

$$\cos\left(l\sqrt{\frac{P}{EI}}\right) = 1 \quad \text{and} \quad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0;$$

whence

$$\left(l\sqrt{\frac{P}{EI}}\right) = 2\pi,$$

and consequently

$$P = \frac{4\pi^2 EI}{l^2},$$

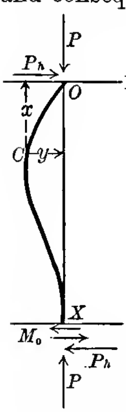


Fig. 95

which is formula (**49**) of the preceding article.

Suppose one end of the column is fixed and the other free to turn, and let P_h denote the horizontal force necessary to keep the free end from lateral movement (Fig. 95). Then the moment at any point C is $M = Py - P_h x$, and the equation of the elastic curve is

$$EI\frac{d^2y}{dx^2} = -Py + P_h x.$$

The general integral of this equation is

$$y = A\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{P_h x}{P},$$

in which A and B are undetermined constants. For $x = 0$ or l, $y = 0$; whence

$$B = 0 \quad \text{and} \quad A = -\frac{P_h l}{P \sin\left(l\sqrt{\frac{P}{EI}}\right)}.$$

For $x = l$, $\frac{dy}{dx} = 0$; whence

$$-\frac{P_h \sqrt{\frac{P}{EI}}\, l \cos\left(l\sqrt{\frac{P}{EI}}\right)}{P \sin\left(l\sqrt{\frac{P}{EI}}\right)} + \frac{P_h}{P} = 0.$$

From the last condition,

$$l\sqrt{\frac{P}{EI}} = \tan\left(l\sqrt{\frac{P}{EI}}\right).$$

This equation is of the form $u = \tan u$, and from this it is found by trial that

$$l\sqrt{\frac{P}{EI}} = 4.49.$$

Consequently,

$$P = \frac{20\,EI}{l^2} = \frac{2\,\pi^2 EI}{l^2}, \text{ approximately.}$$

This equation is of the same form as formula (**50**) of the preceding article, the difference between the numerical constants in the two formulas being due to the approximate nature of the solution given in Article 86.

88. Modification of Euler's formula. It has been found by experiment that Euler's formula applies correctly only to very long columns, and that for short columns or those of medium length it gives a value of P considerably too large.

Very short columns or blocks fail solely by crushing, the tendency to buckle in such cases being practically zero. Therefore, if p denotes the crushing strength of the material and F the area of a cross section, the breaking load for a very short column is $P = pF$.*

* As Euler's formula is based upon the assumption that the column is of sufficient length to buckle sideways, it is evident *à priori* that it cannot be applied to very short columns in which this tendency is practically zero. Thus, in formula (**48**), as l approaches zero P approaches infinity, which of course is inadmissible.

For columns of ordinary length, therefore, the load P must lie somewhere between pF and the value given by Euler's formula. Consequently, to obtain a general formula which shall apply to columns of any length, it is only necessary to express a continuous relation between pF and $\frac{\pi^2 EI}{l^2}$. Such a relation is furnished by the equation

$$(51)\qquad P = \frac{pF}{1 + pF\left(\frac{l^2}{\pi^2 EI}\right)}.$$

For when $l = 0$, $P = pF$, and when l becomes very large P approaches the value $\frac{\pi^2 EI}{l^2}$. Moreover, for intermediate values of l this formula gives values of P considerably less than given by Euler's formula, thus agreeing more closely with experiment.

89. Rankine's formula. Although the above modification of Euler's formula is an improvement on the latter, it does not yet agree closely enough with experiment to be entirely satisfactory. The reason for the discrepancy between the results given by this formula and those obtained from actual tests is that the assumptions upon which the formula is based, namely, that the column is perfectly straight, the material perfectly homogeneous, and the load applied exactly at the centers of gravity of the ends, are never actually realized in practice.

To obtain a more accurate formula, two empirical constants will be introduced into equation (**51**). Thus, for fixed ends, let

$$(52)\qquad P = \frac{gF}{1 + f\left(\frac{l}{t}\right)^2},$$

where f and g are arbitrary constants to be determined by experiment, and t is the least radius of gyration of a cross section of the column. This formula has been obtained in different ways by Gordon, Rankine, Navier, and Schwarz.* Among German writers it is known as

* Rankine's formula can be derived independently of Euler's formula either by assuming that the elastic curve assumed by the center line of the column is a sinusoid, or by assuming that the maximum lateral deflection D at the center of the column is given by the expression $D = \mu\frac{l^2}{b}$, where l is the length of the column, b its least width, and μ an empirical constant.

Schwarz' formula, whereas in English and American text-books it is called **Rankine's formula.**

For $l = 0$, $P = gF$, and since short blocks fail by crushing, g is therefore the ultimate compressive strength of the material.

For different methods of end support Rankine's formula takes the following forms.

Flat ends, (fixed in direction)

$$\frac{P}{F} = \frac{g}{1 + f\left(\frac{l}{t}\right)^2}.$$

Round ends, (direction not fixed)

$$\frac{P}{F} = \frac{g}{1 + 4f\left(\frac{l}{t}\right)^2}.$$

Hinged ends, (position fixed, but not direction)

$$\frac{P}{F} = \frac{g}{1 + 2f\left(\frac{l}{t}\right)^2}.$$

One end flat and the other round,

$$\frac{P}{F} = \frac{g}{1 + 1.78f\left(\frac{l}{t}\right)^2}.$$

90. Values of the empirical constants in Rankine's formula. The values of the empirical constants, f and g, in Rankine's formula have been experimentally determined by Hodgkinson and Christie with the following results.

For hard steel,	$g = 69{,}000$ lb./in.²,	$f = \dfrac{1}{20{,}000}$.
For mild steel,	$g = 48{,}000$ lb./in.²,	$f = \dfrac{1}{30{,}000}$.
For wrought iron,	$g = 36{,}000$ lb./in.²,	$f = \dfrac{1}{36{,}000}$.
For cast iron,	$g = 80{,}000$ lb./in.²,	$f = \dfrac{1}{6{,}400}$.
For timber,	$g = 7{,}200$ lb./in.²,	$f = \dfrac{1}{3{,}000}$.

These constants were determined by experiments upon columns for which $20 < \frac{l}{t} < 200$, and therefore can only be relied upon to

furnish reliable results when the dimensions of the column lie within these limits.

As a factor of safety to be used in applying the formula, Rankine recommended 10 for timber, 4 for iron under dead load, and 5 for iron under moving load.

Problem 170. A solid, round, cast-iron column with flat ends is 15 ft. long and 6 in. in diameter. What load may be expected to cause rupture?

Problem 171. A square wooden post 12 ft. long is required to support a load of 15 tons. With a factor of safety of 10, what must be the size of the post?

Problem 172. Two medium steel Cambria I-beams, No. B 25, weighing 25.25 lb./ft., are joined by lattice work to form a column 25 ft. long. How far apart must the beams be placed, center to center, in order that the column shall be of equal strength to resist buckling in either axial plane?

Problem 173. Four medium steel Cambria angles, No. A 101, 3 in. by 5 in. in size, have their 3-in. legs riveted to a $\frac{3}{4}$-in. plate so as to form an I-shaped built column. How wide must the plate be in order that the column shall be of equal strength to resist buckling in either axial plane?

91. Johnson's parabolic formula. From the manner in which equation (**51**) was obtained and afterward modified by the introduction of the empirical constants f and g, it is clear that Rankine's formula satisfies the requirements for very long or very short columns, while for those of intermediate length it gives the average values of experimental results. A simple formula which fulfills these same requirements has been given by Professor J. B. Johnson, and is called **Johnson's parabolic formula.**

If equation (**52**) is written

$$\frac{P}{F} = p = \frac{g}{1 + f\left(\frac{l}{t}\right)^2},$$

and then y is written for p, and x for $\frac{l}{t}$, Rankine's formula becomes

$$y = \frac{g}{1 + fx^2}.$$

For this cubic equation Johnson substituted the parabola

$$y = \delta - \epsilon x^2,$$

in which x and y have the same meaning as above, and δ and ϵ are empirical constants. The constants δ and ϵ are then so chosen that

the vertex of this parabola is at the elastic limit of the material on the axis of loads (or Y-axis), and the parabola is also tangent to Euler's curve. In this way the formula is made to satisfy the theoretical requirements for very long or very short columns, and for those of intermediate length it is found to agree closely with experiment.

For different materials and methods of end support Johnson's parabolic formulas, obtained as above, are as follows:

Kind of Column	Formula	Limit for Use
Mild steel		
Hinged ends	$\frac{P}{F} = 42{,}000 - .97\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 150$
Flat ends	$\frac{P}{F} = 42{,}000 - .62\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 190$
Wrought iron		
Hinged ends	$\frac{P}{F} = 34{,}000 - .67\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 170$
Flat ends	$\frac{P}{F} = 34{,}000 - .43\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 210$
Cast iron		
Round ends	$\frac{P}{F} = 60{,}000 - \frac{25}{4}\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 70$
Flat ends	$\frac{P}{F} = 60{,}000 - \frac{9}{4}\left(\frac{l}{t}\right)^2$	$\frac{l}{t} \lesseqgtr 120$
Timber (flat ends)		
White pine	$\frac{P}{F} = 2{,}500 - .6\left(\frac{l}{t'}\right)^{2}$*	$\frac{l}{t'} \lesseqgtr 60$
Short-leaf yellow pine	$\frac{P}{F} = 3{,}300 - .7\left(\frac{l}{t'}\right)^2$	$\frac{l}{t'} \lesseqgtr 60$
Long-leaf yellow pine	$\frac{P}{F} = 4{,}000 - .8\left(\frac{l}{t'}\right)^2$	$\frac{l}{t'} \lesseqgtr 60$
White oak	$\frac{P}{F} = 3{,}500 - .8\left(\frac{l}{t'}\right)^2$	$\frac{l}{t'} \lesseqgtr 60$

The limit for use in each case is the value of $x\left(=\frac{l}{t}\right)$ at the point where Johnson's parabola becomes tangent to Euler's curve. For greater values of $\frac{l}{t}$ Euler's formula should therefore be used.

* In the formulas for timber t' is the least lateral dimension of the column.

A graphical representation of the relation between Euler's formula, Rankine's formula, J. B. Johnson's parabolic formula, and T. H. Johnson's straight-line formula (considered in the next article) is given in Fig. 96, for the case of a wrought-iron column with hinged ends.*

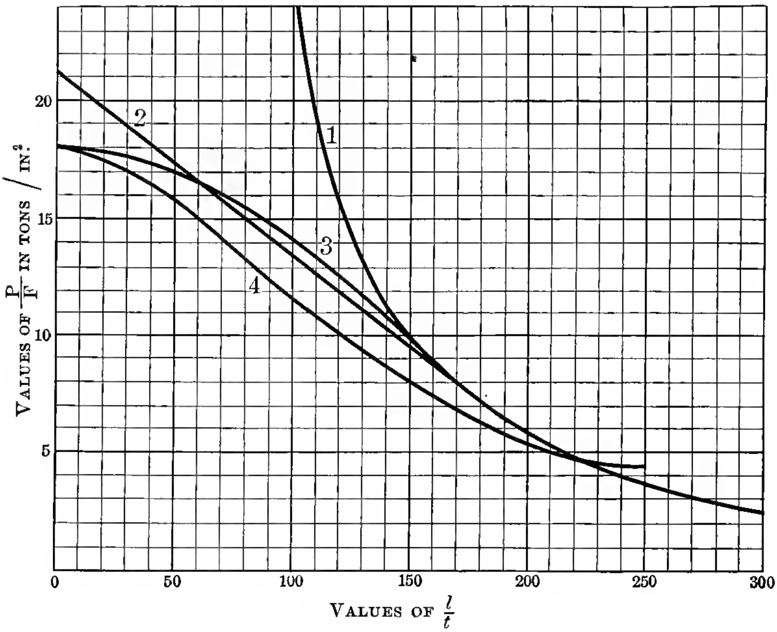


Fig. 96. — Wrought-Iron Column (pin ends)

1, Euler's formula; 2, T. H. Johnson's straight-line formula; 3, J. B. Johnson's parabolic formula; 4, Rankine's formula

Problem 174. A hollow wrought-iron column with flat ends is 20 ft. long, 7 in. internal diameter, and 10 in. external diameter. Calculate its ultimate strength by Rankine's and Johnson's formulas, and compare the results.

Problem 175. Compute the ultimate strength of the built column in Problem 172 by Rankine's and Johnson's formulas, and compare the results.

92. Johnson's straight-line formula. By means of an exhaustive study of experimental data on columns, Mr. Thomas H. Johnson has shown that for columns of moderate length a straight line can be made to fit the plotted results of column tests as exactly as a curve. He has therefore proposed the formula

* For a more extensive comparison of these formulas see Johnson's *Framed Structures*, 8th ed., 1905, pp. 159–171; also *Trans. Amer. Soc. Civ. Eng.*, Vol. XV, pp. 518–536.

$$(53) \qquad \frac{P}{F} = \nu - \sigma \frac{l}{t},$$

or, in the notation of the preceding article,

$$y = \nu - \sigma x,$$

in which ν and σ are empirical constants, this being the equation of a straight line tangent to Euler's curve. This formula has the merit of great simplicity, the only objection to it being that for short columns it gives a value of P in excess of the actual breaking load. The relation of this formula to those which precede is shown in Fig. 96.

The constants ν and σ in formula (**53**) are connected by the relation

$$\sigma = \frac{\nu}{3}\sqrt{\frac{4\,\nu}{3\,n\pi^2 E}},$$

where for fixed ends $n = 1$, for free ends $n = 4$, and for one end fixed and the other free $n = 1.78$.

The table on page 132 gives the special forms assumed by Johnson's straight-line formula for various materials and methods of end support.*

The limit for use in this case is the value of $x\left(= \frac{l}{t}\right)$ for the point at which Johnson's straight line becomes tangent to Euler's curve.

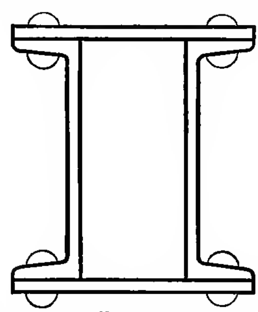


Fig. 97

Problem 176. Compute the ultimate strength of the column in Problem 104 by Rankine's and Johnson's straight-line formulas, and compare the results.

Problem 177. A column 18 ft. long is formed by joining the legs of two Carnegie steel channels, No. C 3, weighing 30 lb./ft., by two plates each 10 in. wide and $\frac{1}{2}$ in. thick, as shown in Fig. 97. Find the safe load for this column by Johnson's straight-line formula, using a factor of safety of 4.

Problem 178. A wrought-iron pipe 10 ft. long, and of internal and external diameter 3 in. and 4 in. respectively, bears a load of 7 tons. What is the factor of safety?

* *Trans. Amer. Soc. Civ. Eng.*, 1886, p. 530.

Kind of Column	Formula	Limit for Use
Hard steel		
Flat ends	$\frac{P}{F} = 80{,}000 - 337\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 158.0$
Hinged ends	$\frac{P}{F} = 80{,}000 - 414\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 129.0$
Round ends	$\frac{P}{F} = 80{,}000 - 534\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 99.9$
Mild steel		
Flat ends	$\frac{P}{F} = 52{,}500 - 179\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 195.1$
Hinged ends	$\frac{P}{F} = 52{,}500 - 220\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 159.3$
Round ends	$\frac{P}{F} = 52{,}500 - 284\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 123.3$
Wrought iron		
Flat ends	$\frac{P}{F} = 42{,}000 - 128\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 218.1$
Hinged ends	$\frac{P}{F} = 42{,}000 - 157\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 178.1$
Round ends	$\frac{P}{F} = 42{,}000 - 203\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 138.0$
Cast iron		
Flat ends	$\frac{P}{F} = 80{,}000 - 438\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 121.6$
Hinged ends	$\frac{P}{F} = 80{,}000 - 537\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 99.3$
Round ends	$\frac{P}{F} = 80{,}000 - 693\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 77.0$
Oak		
Flat ends	$\frac{P}{F} = 5{,}400 - 28\,\frac{l}{t}$	$\frac{l}{t} \overline{\gtrless} 128.1$

93. Cooper's modification of Johnson's straight-line formula. In his standard bridge specifications, Theodore Cooper has adopted Johnson's straight-line formulas, modifying them by the introduction of a factor of safety. Thus, for medium steel, Cooper specifies that the following formulas shall be used in calculating the *safe* load.

For chords

$$\begin{cases} \dfrac{P}{F} = 8{,}000 - 30\,\dfrac{l}{t} \text{ for live load stresses,} \\ \dfrac{P}{F} = 16{,}000 - 60\,\dfrac{l}{t} \text{ for dead load stresses.} \end{cases}$$

For posts

$$\begin{cases} \dfrac{P}{F} = 7{,}000 - 40\,\dfrac{l}{t} \text{ for live load stresses,} \\ \dfrac{P}{F} = 14{,}000 - 80\,\dfrac{l}{t} \text{ for dead load stresses,} \\ \dfrac{P}{F} = 10{,}000 - 60\,\dfrac{l}{t} \text{ for wind stresses.} \end{cases}$$

For lateral struts

$$\frac{P}{F} = 9{,}000 - 50\,\frac{l}{t} \text{ for initial stresses.}$$

By initial stress in the last formula is meant the stress due to the adjustment of the bridge members during construction.

Problem 179. What must be the size of a square steel strut 8 ft. long, to transmit a load of 5 tons with safety?

Problem 180. Design a column 16 ft. long to be formed of two channels joined by two plates and to support a load of 20 tons with safety.

Problem 181. Using Cooper's formula for live load, design the inclined end post of a bridge which is 25 ft. long and bears a load of 30 tons, the end post to be composed of four angles, a top plate, and two side plates.

94. Beams of considerable depth. When narrow beams of considerable depth are subjected to compression, as, for example, in a deck plate girder bridge, the strain is similar to that in a column. For a narrow, deep beam the inertia ellipse is greatly elongated, and consequently the radius of gyration relative to a line forming a small angle with the horizontal is considerably less than the semi-major axis of the ellipse. Therefore, if the beam is thrown slightly out of the vertical by the unequal settling of its supports, or by any other cause, such inclination results in a notable decrease in its resistance. Since it is impossible to make allowances for such accidental reductions of strength, beams of great depth or very thin web should be avoided.

95. Eccentrically loaded columns. In a column carrying an eccentric load, as, for example, a column carrying a load on a bracket, or the post of a crane, there is a definite amount of bending stress due to the eccentricity of the load in addition to the column stress. As the nature of column stress is such that it is impossible to determine its amount, the simplest method of handling a problem of this

kind is to determine its relative security against failure as a column and failure by bending. That is to say, first determine its factor of safety against failure as a column under the given column load. Then consider it as a beam and find the equivalent bending moment which would give the same factor of safety. Finally, combine this equivalent bending moment with that due to the eccentric load, and calculate the unit stress from the ordinary beam formulas.

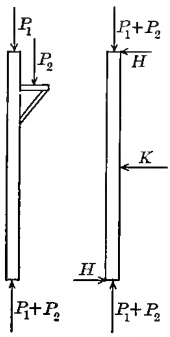


FIG. 98

To illustrate the method, suppose that a column 18 ft. long is composed of two 12-in. I-beams, each weighing 40 lb./ft., and carries a column load of 20 tons at its upper end and also an eccentric load of 10 tons with eccentricity 2 ft., as shown in Fig. 98. Assuming that the column has flat ends, and using Johnson's straight-line formula $P = F\left(52{,}500 - 179\frac{l}{t}\right)$, the factor of safety against column failure is

$$\text{Factor of safety} = \frac{F\left(52{,}500 - 179\frac{l}{t}\right)}{60{,}000}$$
$$= \frac{2(11.76)(52{,}500 - 179(47.3))}{60{,}000} = 17.3.$$

Now consider the column as a beam, and find the equivalent central load K corresponding to the factor of safety just found, namely 17.3. The maximum moment in a simple beam bearing a concentrated load K at the center is $M = \frac{Kl}{4}\cdot$ Hence from the beam formula $M = \frac{pI}{e}$ we have $\frac{Kl}{4} = \frac{pI}{e}$, whence

$$K = \frac{4pI}{le}.$$

Assuming the ultimate strength of the material to be 60,000 lb./in.^{2}, we have

$$p = \frac{60{,}000}{17.3} \text{ lb./in.}^2, \qquad I = 2(245.9) \text{ in.}^4,$$
$$l = 216 \text{ in.}, \qquad e = 6 \text{ in.},$$

and inserting these values, the equivalent load K is found to be

$$K = \frac{4 \times 60{,}000 \times 491.8}{17.3 \times 216 \times 6} = 5220 \text{ lb.}$$

Now the eccentric load P_2 acting parallel to the axis of the column produces the same bending effect as a horizontal reaction H at either end, where $Hl = P_2 d$. The bending moment at the center, due to a reaction H perpendicular to the axis of the beam, is, however, $\frac{Hl}{2}$. Hence the total equivalent moment at the center now becomes

$$\begin{aligned} M &= \frac{Kl}{4} + \frac{Hl}{2} = \frac{Kl}{4} + \frac{P_2 d}{2} \\ &= \frac{5220 \times 216}{4} + \frac{20{,}000 \times 24}{2} \\ &= 521{,}880 \text{ in. lb.} \end{aligned}$$

Consequently, the maximum unit stress in the member becomes

$$\begin{aligned} p &= \frac{M}{S} \\ &= \frac{521{,}880}{81.96} = 6367 \text{ lb./in.}^2, \end{aligned}$$

which corresponds to a factor of safety of about 9.

If this factor of safety is larger than desired, assume a smaller I-beam and repeat the calculations.

A method substantially equivalent to the above is to assume that the stress in a column is represented by the empirical factor in the column formula used. Thus for a short block, the actual compressive stress p is given by the relation $P = pF$, whereas in the column formula used above, namely $P = F\left(52{,}500 - 179\frac{l}{t}\right)$, the stress p is replaced by the empirical factor $52{,}500 - 179\frac{l}{t}$. Consequently, the fraction

$$\frac{52{,}500 - 179\frac{l}{t}}{u_c},$$

where u_c denotes the ultimate compressive strength of the material, represents the reduction in strength of the member due to its slimness and method of loading; or, what amounts to the same thing, the equivalent unit stress in the column is

$$p_e = \frac{P}{F}\left(\frac{u_c}{52{,}500 - 179\frac{l}{t}}\right).$$

Applying this method to the numerical problem given above, we have $F = 23.52$,

$$\frac{l}{t} = 47.3, \text{ and } \frac{u_c}{52{,}500 - 179\frac{l}{t}} = \frac{60{,}000}{52{,}500 - 179 \times 47.3} = 1.36.$$

Hence the equivalent stress in the column is

$$p_e = \frac{30 \times 2000}{23.52} \times 1.36 = 3470 \text{ lb./in.}^2$$

Also, the bending stress, produced by the eccentricity of the load, is

$$p = \frac{P_2 d}{S} = \frac{240{,}000}{81.96} = 2928 \text{ lb./in.}^2$$

Consequently, by this method, the total stress in the column is found to be

$$3470 + 2928 = 6398 \text{ lb./in.}^2$$

If a formula of the Rankine-Gordon type is used, namely,

$$\frac{P}{F} = \frac{g}{1 + f\left(\frac{l}{t}\right)^2},$$

the equivalent stress p_e in the column, due to the given load P, is

$$p_e = \frac{P}{F}\left[\frac{u_c\left(1 + f\left(\frac{l}{t}\right)^2\right)}{g}\right],$$

where u_c denotes the ultimate compressive strength of the material, as above.

EXERCISES ON CHAPTER V

Problem 182. A strut 16 ft. long, fixed rigidly at both ends, is needed to support a load of 80,000 lb. It is to be composed of two pairs of angles united with a single line of $\frac{1}{4}$-in. lattice bars along the central plane. Determine the size of the angles for a factor of safety of 5.

Note that the angles must be spread $\frac{1}{2}$ in. to admit the latticing.

Problem 183. For short posts or struts, such as are ordinarily used in building construction, it is customary to figure the safe load as 12,000 lb./in.^{2} of cross-section area for lengths up to 90 times the radius of gyration, i.e. for $\frac{l}{r} \leqq 90$. To what factor of safety does this correspond, using Johnson's straight-line formula?

Problem 184. The posts used to support a girder in a building are 8 in. × 8 in. timbers 8 ft. long. Find the diameter of a solid cast-iron column of equal strength.

If a wrought-iron pipe 4 in. in external diameter is used, what must be its thickness to be equally safe?

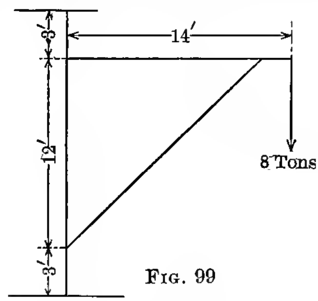


Fig. 99

Problem 185. At what ratio of diameter to length would a round mild steel strut have the same tendency to crush as to buckle?

Problem 186. A load of 100 tons is carried jointly by three cast-iron columns 20 ft. long. What saving in material will be effected by using a single column instead of three, the factor of safety to be 15 in both cases?

Problem 187. Determine the proper size for a hard-steel piston rod 48 in. long for a piston 18 in. in diameter and a steam pressure of 80 lb./in.^{2} Consult table for proper factor of safety.

Problem 188. The side rod of a locomotive is 9 ft. long between centers, 4 in. deep, and 2 in. wide. The estimated thrust in the rod is 12 tons, and the transverse inertia and gravity load 20 lb. per inch of length. Determine the factor of safety.

Problem 189. The vertical post of a crane, sketched in Fig. 99, is to be made of a single I-beam. The post is pivoted at both ends so as to revolve about its axis. Find the size of I-beam required for factor of safety of 4, and for dimensions and loading as shown in the figure.

CHAPTER VI

TORSION

96. Circular shafts. When a uniform circular shaft, such as shown in Fig. 100, is twisted by the application of moments of opposite signs to its ends, every straight line AB parallel to its axis is deformed into part of a helix, or screw thread, AC. The strain in this case is one of pure shear and is called **torsion,** as mentioned in Article 37. The angle ϕ is called the **angle of shear** (compare Article 33), and is proportional to the radius BD of the shaft. The angle θ is called the **angle of twist**, and is proportional to the length AB of the shaft.

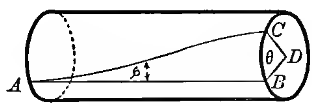


Fig. 100

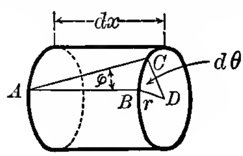


Fig. 101

97. Maximum stress in circular shafts. Consider a section of length dx cut from a circular shaft by planes perpendicular to its axis (Fig. 101). Let $d\theta$ denote the angle of twist for this section. Then, since the angle of twist is proportional to the length of the shaft, $d\theta : \theta = dx : l$; whence

$$d\theta = \theta \frac{dx}{l}.$$

Also, if ϕ and $d\theta$ are expressed in circular measure,

$$BC = \phi \cdot AB = \phi dx,$$

and

$$BC = d\theta \cdot BD = r d\theta.$$

Therefore

$$\phi = \frac{r d\theta}{dx} = r \frac{\theta}{l}.$$

From Hooke's law (Article 33), $\frac{q}{\phi} = G$. Hence

$$q = G\phi = \frac{Gr\theta}{l}. \tag{54}$$

Therefore q is proportional to r; that is to say, the unit shear is proportional to its distance from the center, being zero at the center and attaining its maximum value at the circumference.

If q' denotes the intensity of the shear at the circumference and a denotes the radius of the shaft, then the shear q at a distance r from the center is given by the formula

$$q = \frac{q'r}{a}.$$

Let M_t denote the external twisting moment. Then, since M_t must be equal to the internal moment of resistance,

$$M_t = \int qrdF = \frac{q'}{a}\int r^2 dF = \frac{q'I_p}{a},$$

where I_p is the polar moment of inertia of the section.

For a solid circular shaft of diameter D, $I_p = \frac{\pi D^4}{32}$, and consequently

$$q' = \frac{M_t a}{I_p} = \frac{16\, M_t}{\pi D^3}. \tag{55}$$

For a hollow circular shaft of external diameter D and internal diameter d, $I_p = \frac{\pi}{32}(D^4 - d^4)$, and hence

$$q' = \frac{\overset{16}{\cancel{8}}\, M_t D}{\pi (D^4 - d^4)}. \tag{56}$$

98. Angle of twist in circular shafts. From equation (54),

$$\theta = \frac{ql}{Gr} = \frac{q'l}{Ga}.$$

Therefore, for a solid circular shaft, from equation (55),

$$\theta = \frac{32\, M_t l}{\pi D^4 G}. \tag{57}$$

and for a hollow circular shaft, from equation (56),

$$\theta = \frac{32\, M_t l}{\pi G (D^4 - d^4)}. \tag{58}$$

If M_t is known and θ can be measured, equations (57) and (58) can be used for determining G. If G is known and θ measured, these equations can be used for finding M_t; in this way the horse power which a rotating circular shaft is transmitting can be determined.

Problem 190. A steel wire 20 in. long and .182 in. in diameter is twisted by a moment of 20 in. lb. The angle of twist is then measured and found to be $\theta = 18° 31'$. What is the value of G determined from this experiment?

Problem 191. If the angle of twist for the wire in Problem 190 is $\theta = 40°$, how great is the torsional moment acting on the wire?

99. Power transmitted by circular shafts. Let H be the number of horse power transmitted by the shaft, and n the number of revolutions it makes per minute. Then, if q is the force acting on a particle at a distance r from the center, the moment of this force is qr, and consequently the total moment transmitted by the shaft is $M_t = \int qrdF$. Also, the distance traveled by q in one minute is $2\,\pi rn$, and therefore the total work transmitted by the shaft is

$$W = \int 2\,\pi rnqdF.$$

Since 1 horse power = 33,000 ft. lb./min. = 396,000 in. lb./min., the total work done by the shaft is

$$W = 396{,}000\ H \text{ in. lb./min.}$$

Therefore

$$2\,\pi n \int rqdF = W = 396{,}000\ H,$$

or

$$2\,\pi n M_t = 396{,}000\ H;$$

whence

$$M_t = \frac{396{,}000\ H}{2\,\pi n} = 63{,}030\,\frac{H}{n} \text{ in. lb.}$$

Therefore, if it is required to find the diameter D of a solid circular shaft which shall transmit a given horse power H with safety, then from equation (55),

$$q' = \frac{16\, M_t}{\pi D^3} = \frac{321{,}000\ H}{nD^3};$$

whence

$$D = 68.5 \sqrt[3]{\frac{H}{nq'}}. \tag{59}$$

As safe values for the maximum unit shear q' Ewing recommends 9000 lb./in.^{2} for wrought iron, 13,500 lb./in.^{2} for steel, and 4500 lb./in.^{2} for cast iron.* Inserting these values of q' in formula (59), it becomes

$$D = \mu \sqrt[3]{\frac{H}{n}}, \tag{60}$$

where for steel $\mu = 2.88$, for wrought iron $\mu = 3.29$, and for cast iron $\mu = 4.15$.

Expressed in kilowatts instead of horse power, this formula becomes

$$D = \rho \sqrt[3]{\frac{K.W.}{n}},$$

where for steel $\rho = 3.175$, for wrought iron $\rho = 3.627$, and for cast iron $\rho = 4.576$.

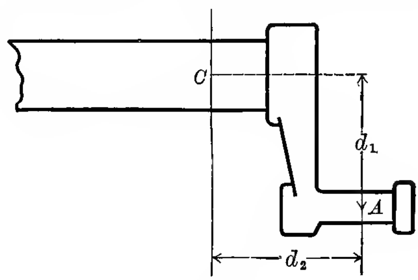


FIG. 102

Problem 192. Find the diameter of a solid wrought-iron circular shaft which is required to transmit 150 H.P. at a speed of 60 revolutions per minute.

Problem 193. A steel shaft is required to transmit 300 H.P. at a speed of 200 revolutions per minute, the maximum moment being 40 per cent greater than the average. Find the diameter of the shaft.

Problem 194. Under the same conditions as in Problem 193, find the inside diameter of a hollow circular shaft whose outside diameter is 6 in. Also compare the amount of metal in the solid and hollow shafts.

Problem 195. How many H.P. can a hollow circular steel shaft of 15 in. external diameter and 11 in. internal diameter transmit at a speed of 50 revolutions per minute, if the maximum allowable unit stress is not to exceed 12,000 lb./in.^{2}?

100. Combined bending and torsion. When a shaft transmits power by means of a crank or pulley, it is subjected to combined bending and torsion. For example, if a force P acts at a point A in the crank pin shown in Fig. 102, the bending moment at any point C of the shaft is $M_b = Pd_2$, and the torsional moment at C is $M_t = Pd_1$.

* Ewing, *The Strength of Materials*, p. 190.

Therefore, if D is the diameter of the shaft at C, the normal stress on the extreme fiber due to bending is

$$p = \frac{32\,M_b}{\pi D^3},$$

and the shearing stress on the extreme fiber due to torsion is

$$q = \frac{16\,M_t}{\pi D^3}.$$

There is also a shearing stress of amount P distributed over the cross section through C, but since it is zero at the outer fiber, it does not enter into this calculation.

From Article 26, the values of the principal stresses are

$$p'_{\substack{\max\\ \min}} = \frac{p}{2} \pm \tfrac{1}{2}\sqrt{4\,q^2 + p^2},$$

and from Article 28, the maximum or minimum shear is

$$q'_{\substack{\max\\ \min}} = \pm\,\tfrac{1}{2}\sqrt{4\,q^2 + p^2}.$$

Inserting in these expressions the values of p and q obtained above, the principal stresses and the maximum or minimum shear are, in the present case,

$$p'_{\substack{\max\\ \min}} = \frac{16}{\pi D^3}\left(M_b \pm \sqrt{M_b^2 + M_t^2}\right) \text{ (called Rankine's formula),}$$

$$q'_{\substack{\max\\ \min}} = \pm\frac{16}{\pi D^3}\sqrt{M_b^2 + M_t^2} \text{ (called Guest's formula).}$$

The equivalent stress may also be found. Thus, from equation (**15**), Article 36, assuming $m = 3\frac{1}{3}$, its value is found to be

$$p_e = \frac{16}{\pi D^3}\left[.7\,M_b \pm 1.3\sqrt{M_b^2 + M_t^2}\right] \text{ (called St. Venant's formula).}$$

It is evident that St. Venant's formula is simply a refinement on Rankine's, as both give the principal normal stresses, whereas Guest's formula is essentially different, since it gives shear. Hence in designing members subjected to both bending and torsion, try both Guest's and Rankine's (or St. Venant's) formulas with the same factor of safety, and then use whichever gives the larger dimensions to the construction.

Problem 196. A steel shaft 5 in. in diameter is driven by a crank of 12-in. throw, the maximum thrust on the crank being 10 tons. If the outer edge of the shaft bearing is 11 in. from the center of the crank pin, what is the equivalent stress in the shaft at this point?

Problem 197. A steel shaft 10 ft. long between bearings and 4 in. in diameter carries a pulley 14 in. in diameter at its center. If the tension in the belt on this pulley is 250 lb., and the shaft makes 80 revolutions per minute, what is the maximum stress in the shaft and how many H.P. is it transmitting?

***101. Resilience of circular shafts.** In Article 74 the resilience of a body was defined as the internal work of deformation. For a solid circular shaft this internal work is

$$W = \tfrac{1}{2} M_t \theta,$$

where M_t is the external twisting moment and θ is the angle of twist. From equation (54), $\theta = \dfrac{ql}{Gr} = \dfrac{q'l}{Ga}$, and from equation (55), $M = \dfrac{\pi D^3 q'}{16}$. Therefore the total resilience of the shaft is

$$W = \frac{1}{2} M_t \theta = \frac{\pi D^2 q'^2 l}{16 G},$$

and consequently the mean resilience per unit of volume is

$$W_1 = \frac{W}{V} = \frac{q'^2}{4G}.$$

102. Non-circular shafts. The above investigation of the distribution and intensity of torsional stress applies only to shafts of circular section. For other forms of cross section the results are entirely different, each form having its own peculiar distribution of stress.

For any form of cross section whatever, the stress at the boundary must be tangential. For if the stress is not tangential, it can be resolved into two components, one tangential and the other normal to the boundary; and in Article 23 it was shown that such a normal component would necessitate forces parallel to the axis of the shaft, which are excluded by hypothesis.

Since the stress at the boundary must be tangential, the circular section is the only one for which the stress is perpendicular to a radius vector. Therefore the circular section is the only one to which the above development applies, and consequently is the only form of

* For a brief course the remainder of this chapter may be omitted.

cross section for which Bernoulli's assumption holds true. That is to say, *the circular section is the only form of cross section which remains plane under a torsional strain.*

The subject of the distribution of stress in non-circular shafts has been investigated by St. Venant, and the results of his investigations are summarized below (Articles 103–106 inclusive).

103. Elliptical shaft. For a shaft the cross section of which is an ellipse of semi-axes a and b, the maximum stress occurs at the ends of the *minor* axis, instead of at the ends of the major axis, as might be expected. The unit stress at the ends of the minor axis is given by the formula

$$q_{\max} = \frac{2\,M_t}{\pi a b^2},$$

and the angle of twist *per unit of length* is

$$\theta_1 = \frac{M_t(a^2 + b^2)}{\pi a^3 b^3 G}.$$

The total angle of twist for an elliptical shaft of length l is therefore

$$\theta = \theta_1 l = \frac{M_t(a^2 + b^2)l}{\pi a^3 b^3 G}.$$

Problem 198. The semi-axes of the cross section of an elliptical shaft are 3 in. and 5 in. respectively. What is the diameter of a circular shaft of equal strength?

104. Rectangular and square shafts. For a shaft of rectangular cross section the maximum stress occurs at the centers of the longer sides, its value at these points being

$$(61) \qquad q_{\max} = \frac{6\,M_t}{hb\sqrt{h^2 + b^2}}\left(.68 + .45\frac{h}{b}\right),$$

in which h is the longer and b the shorter side of the rectangle. The angle of twist per unit of length is, in this case,

$$\theta_1 = 3.57\,\frac{M_t(h^2 + b^2)}{Gh^3b^3}.$$

For a square shaft of side b these formulas become

$$(62) \qquad q_{\max} = 4.8\,\frac{M_t}{b^3},$$

and

$$\theta_1 = 7.14\,\frac{M_t}{Gb^4}.$$

The value of q for a square shaft found from this equation is about 15 per cent greater than if the formula $q = \frac{Mr}{I_p}$ was used, and the torsional rigidity is about .88 of the torsional rigidity of a circular shaft of equal sectional area.

Problem 199. An oak beam 6 in. square projects 4 ft. from a wall and is acted upon at the free end by a twisting moment of 25,000 ft. lb. How great is the angle of twist?

105. Triangular shafts. For a shaft whose cross section is an equilateral triangle of side c,

$$q_{\max} = 20 \frac{M_t}{c^3},$$

and the angle of twist per unit of length is

$$\theta_1 = \frac{M_t}{.6\, G I_p}.$$

The torsional rigidity of a triangular shaft is therefore .73 of the torsional rigidity of a circular shaft of equal sectional area.

106. Angle of twist for shafts in general. The formula for the angle of twist per unit of length for circular and elliptical shafts can be written

$$\theta_1 = \frac{4\pi^2 M_t}{G} \cdot \frac{I_p}{F^4},$$

in which I_p is the polar moment of inertia of a cross section about its center, and F is the area of the cross section. This formula is rigorously true for circular and elliptical shafts, and St. Venant has shown that it is approximately true whatever the form of cross section.

Problem 200. Compare the angle of twist given by St. Venant's general formula with the values given by the special formulas in Articles 103, 104, and 105.

Problem 201. Find the angle of twist in Problem 193.

Problem 202. Find the angle of twist in Problem 194, and compare it with the angle of twist for the solid shaft in Problem 201.

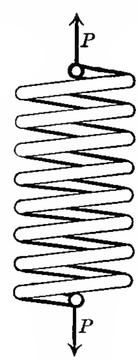


FIG. 103

107. Helical springs. The simplest form of a helical, or spiral, spring is formed by wrapping a wire upon a circular cylinder, the form of such a spring being that of a screw thread. Let r be the radius of the coil and a the radius of the wire, and let the spring be either compressed or extended by

two forces P acting in the direction of the axis of the cylinder (Fig. 103). Then the bending moment at any point of the spring is $M = Pr$. If the radius r of the coil is large in comparison with the diameter of the wire, and if the spring is closely wound, the plane of the external moment M is very nearly perpendicular to the axis of the helix, and consequently the bending strain can be assumed to be zero in comparison with the torsional strain. Under this assumption the maximum stress is found, from equation (55), to be

$$q' = q_{\max} = \frac{2\,M_t}{\pi a^3} = \frac{2\,Pr}{\pi a^3}.$$

Similarly, the maximum stress in a spring of square or rectangular cross section can be found by substituting $M = Pr$ in equations (61) and (62).

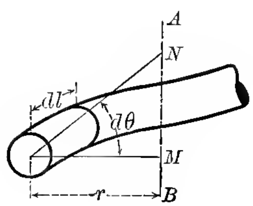


FIG. 104

To find the amount by which the spring is extended or compressed, let $d\theta$ be the angle of twist for an element of the helix of length dl. Then (Fig. 104), if AB is the axis of the spring, a point M in this axis in the same horizontal plane with the element dl is displaced vertically an amount $MN = rd\theta$ in the direction of the axis. Therefore the total axial compression or extension D of the spring is the sum of all the infinitesimal displacements $rd\theta$ for every element dl; whence

$$D = \int r d\theta.$$

From equation (57), $\theta = \dfrac{2\,Ml}{\pi a^4 G} = \dfrac{2\,Prl}{\pi a^4 G}$.

Therefore $d\theta = \dfrac{2\,Pr}{\pi a^4 G}\,dl$, and consequently

$$D = \int_0^l \frac{2\,Pr^2}{\pi a^4 G}\,dl = \frac{2\,Pr^2}{\pi a^4 G}\int_0^l dl = \frac{2\,Pr^2 l}{\pi a^4 G},$$

in which l is the length of the helix.

If n denotes the number of turns of the helix, then, under the above assumption that the slope of the helix is small, $l = 2\pi rn$ approximately, and hence

$$D = \frac{4\,Pr^3n}{a^4G}, \text{ approximately.}$$

The resilience W of the spring is equal to one half the product of the force P multiplied by the axial extension or compression of the spring. Hence

$$W = \frac{1}{2}PD = \frac{P^2r^2l}{\pi a^4G}.$$

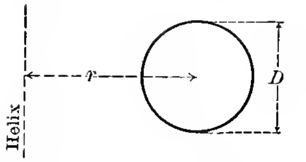
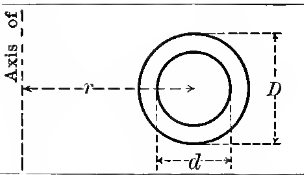
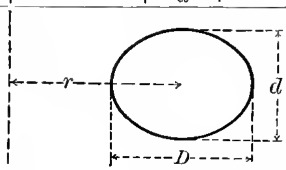
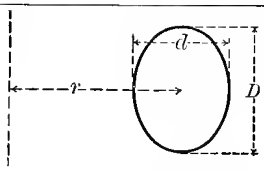
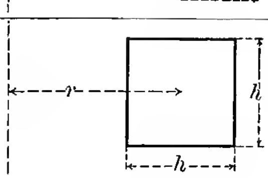
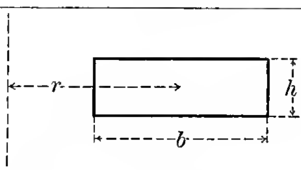
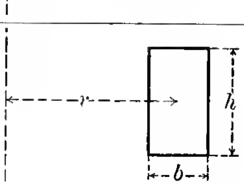
Problem 203. A helical spring is composed of 20 turns of steel wire .258 in. in diameter, the diameter of the coil being 3 in. If the spring is compressed by a force of 50 lb., what is the maximum stress in the spring, its axial compression, and its resilience?

108. General theory of spiral springs. The general theory of the cylindric spiral spring subjected to both axial load and torque has many important applications in physics and engineering, as, for example, in the helical-spring transmission dynamometer now coming into general use by reason of its ability to measure power without absorbing it.

To analyze the most general case, suppose that an axial load P is applied to the spring and also a torque M_t, the positive direction of P being chosen as that which will produce elongation of the spring, and the positive direction of M_t as that which will increase the number of coils. Also let

x = axial elongation of spring,
ϕ = angular rotation of spring,
r = radius of coils, i.e. distance from center of wire to axis,
α = angle of spiral (pitch angle),
l = length of wire,
A = torsional rigidity = $\dfrac{\text{torque}}{\text{angle of twist per unit length}}$,
B = flexural rigidity = $\dfrac{\text{bending moment}}{\text{flexure per unit length of wire}}$.

Note that if E = Young's modulus and G = shear modulus (modulus of rigidity), then $B = EI$, where I denotes the static moment of inertia of a cross section of the wire with respect to its neutral axis, but that A is *not* equal to GI_p, where I_p denotes the polar moment of inertia of a cross section of the wire. The true values of A for various cross sections, however, may be found from Articles 105–108 in connection with the above definition, and are summarized in the following table:

SHAPE	TORSIONAL RIGIDITY A	FLEXURAL RIGIDITY B
	$A = \frac{T}{\theta_1} = \frac{\pi G d^4}{32}$ Eq. 57, Art. 98	$B = \frac{\pi d^4 E}{64}$
	$A = \frac{\pi G}{32}(D^4 - d^4)$ Eq. 58, Art. 98	$B = \frac{\pi E}{64}(D^4 - d^4)$
	$A = \frac{\pi G}{16} \cdot \frac{D^3 d^3}{D^2 + d^2}$ Art. 103	$B = \frac{\pi E}{64} d D^3$
	$A = \frac{\pi G}{16} \frac{D^3 d^3}{D^2 + d^2}$ Art. 103	$B = \frac{\pi E}{64} D d^3$
	$A = \frac{G h^4}{7.14}$ Art. 104	$B = \frac{E h^4}{12}$
	$A = \frac{G}{3.57} \frac{b^3 h^3}{b^2 + h^2}$ Art. 104	$B = \frac{E h b^3}{12}$
	$A = \frac{G}{3.57} \frac{b^3 h^3}{b^2 + h^2}$ Art. 104	$B = \frac{E h b^3}{12}$

Now consider any cross section GOF of the wire (Fig. 105) and draw the Y-axis through O parallel to the axis of the helix, and the X-axis at right angles to OY and tangent to the cylinder on which the helix is wound. Also draw another pair of rectangular axes in the XOY-plane, namely OV tangent to the helix and OU normal to it.

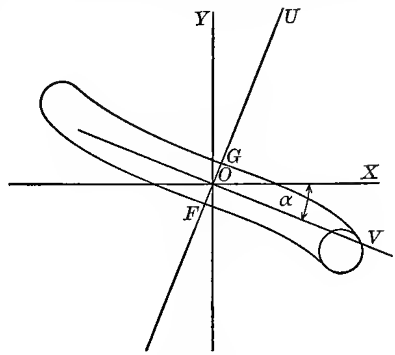


FIG. 105

The axial load P produces a moment M_b about OX of amount $M_b = Pr$, while the torque M_t acts about the axis OY. Represent each of these moments by a vector, that is, a single line, the length of which represents the numerical amount of the moment, and the direction of which is the same as that in which a right-handed screw would advance if revolved in the direction indicated by the given moment. In the present case the torque M_t causes revolution about the Y-axis and is therefore represented by a vector laid off along this axis and pointing upward; while similarly the axial load P produces revolution about the X-axis and is represented by a vector M_b laid off along OX and pointing to the left (Fig. 106).

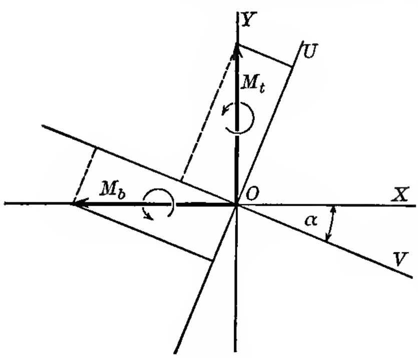


FIG. 106

Now in order to obtain the bending and twisting moments acting on the wire, these vectors must be resolved in the directions OU and OV. Hence

$$\text{Moment about } OV \text{ (torsional moment)} = M_t \sin\alpha + M_b \cos\alpha,$$
$$\text{Moment about } OU \text{ (bending moment)} \;\; = M_t \cos\alpha - M_b \sin\alpha.$$

Consequently, from the above definitions of torsional and flexural rigidity, we have

$$\text{Angle of twist per unit of length} = \frac{M_t \sin\alpha + M_b \cos\alpha}{A},$$

$$\text{Flexure per unit of length} \qquad = \frac{M_t \cos\alpha - M_b \sin\alpha}{B}.$$

To obtain the axial deflection of the spring and its angular rotation about its axis, these quantities must next be projected back on the X- and Y- axes. Making this projection (Fig. 107), we have

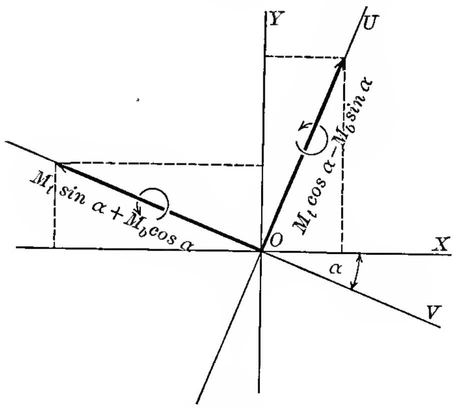


FIG. 107

Rotation about vertical axis OY per unit length of wire

$$= \frac{M_t \cos\alpha - M_b \sin\alpha}{B} \cos\alpha + \frac{M_t \sin\alpha + M_b \cos\alpha}{A} \sin\alpha,$$

Rotation about horizontal axis OX per unit length of wire

$$= \frac{M_t \sin\alpha + M_b \cos\alpha}{A} \cos\alpha - \frac{M_t \cos\alpha - M_b \sin\alpha}{B} \sin\alpha.$$

Multiplying each of these expressions by the length of the wire l, and simplifying, we have finally

$$\phi = M_b l \sin\alpha \cos\alpha \left(\frac{1}{A} - \frac{1}{B}\right) + M_t l \left(\frac{\cos^2\alpha}{B} + \frac{\sin^2\alpha}{A}\right),$$

$$x = M_b r l \left(\frac{\cos^2\alpha}{A} + \frac{\sin^2\alpha}{B}\right) + M_t l r \sin\alpha \cos\alpha \left(\frac{1}{A} - \frac{1}{B}\right).$$

Special Case I. Spirals very flat.

In case the spirals are very flat, α may be as sumed to be zero, and the above expressions then reduce to

$$\phi = \frac{M_t l}{B}, \qquad x = \frac{M_b l r}{A} = \frac{P l r^2}{A},$$

Special Case II. Rotation of ends prevented, i.e. $\phi = 0$.

In this case first find M_t in terms of M_b from the equation $\phi = 0$; then substitute this value of M_t in the x equation and find x in terms of M_b.

Problem 204. Assume $\phi = 0$ and $\alpha = 45°$. Find x.

Solution. Substituting these numerical values of ϕ and α in the expression

$$\phi = M_b l \sin\alpha\cos\alpha\left(\frac{1}{A} - \frac{1}{B}\right) + M_t l\left(\frac{\cos^2\alpha}{B} + \frac{\sin^2\alpha}{A}\right)$$

and solving the resulting equation for M_t, we have

$$M_t = M_b\frac{A - B}{A + B}.$$

Substituting this value of M_t in the equation

$$x = M_b r l\left(\frac{\cos^2\alpha}{A} + \frac{\sin^2\alpha}{B}\right) + M_t l r\sin\alpha\cos\alpha\left(\frac{1}{A} - \frac{1}{B}\right),$$

we have finally

$$x = \frac{2\,M_b l r}{A + B} = \frac{2\,P l r^2}{A + B}.$$

Special Case III. Axial elongation prevented, i.e. $x = 0$.

This case applies to the helical-spring transmission dynamometer mentioned above. In this case first find M_b in terms of M_t from the equation $x = 0$, and then substitute this value of M_b in the ϕ equation and find ϕ in terms of M_t.

Problem 205. A helical-spring transmission dynamometer is made of $15\frac{3}{4}$ turns of $\frac{1}{4}$-in. steel wire, the mean diameter of the coil being $1\frac{1}{4}$ in., and there being 2 turns of wire per inch. Calculate the torque required to produce an angular deflection of 25°.

Solution. From the table the constants A and B for a circular cross section are $A = \frac{\pi G d^4}{32}$ and $B = \frac{\pi E d^4}{64}$. Inserting these values in the equation $x = 0$, and assuming the relation between the elastic moduli to be $G = \frac{2}{5}E$, the result is

$$M_b = M_t\frac{\sin\alpha\cos\alpha}{5\cos^2\alpha + 4\sin^2\alpha}.$$

From the given dimensions it is found that $\alpha = 7° 15.4'$, and consequently

$$M_b = .02513\, M_t.$$

Inserting this value in the ϕ equation and also making $d = .25$, $\phi = \frac{25\,\pi}{180}$, $l = 62.35$, and $E = 30{,}000{,}000$, the result of solving for M_t is

$$M_t = 40.06 \text{ in. lb.}$$

EXERCISES ON CHAPTER VI

Problem 206. Find the diameter of a structural-steel engine shaft to transmit 900 H.P. at 75 R.P.M. with a factor of safety of 10.

Problem 207. Find the factor of safety for a wrought-iron shaft 5 in. in diameter which is transmitting 60 H.P. at 125 R.P.M.

Problem 208. A structural-steel shaft is 60 ft. long and is required to transmit 500 H.P. with a factor of safety of 8 and to be of sufficient stiffness so that the angle of torsion shall not exceed .5° per foot of length. Find its diameter.

Problem 209. Under the same conditions as in Problem 208 find the size of a hollow shaft if the external diameter is twice the internal.

Problem 210. A hollow wrought-iron shaft 9 in. in external diameter and 2 in. thick is required to transmit 600 H.P. with a factor of safety of 10. At what speed should it be run?

Problem 211. A horizontal steel shaft 4 in. in diameter and 10 ft. long between centers of bearings carries a pulley weighing 300 lb. and 14 in. in diameter. The belt on the pulley has a tension of 50 lb. on the slack side and 175 lb. on the driving side. Find the maximum combined stress in the shaft.

Problem 212. An overhung steel crank carries a maximum thrust on the crank pin of 2 tons. Length of crank 9 in.; distance from center of pin to center of bearing 5 in. Determine the size of crank and shaft for a factor of safety of 5.

Problem 213. A propeller shaft 9 in. in diameter transmits 1000 H.P. at 90 R.P.M. If the thrust on the screw is 12 tons, determine the maximum stress in the shaft.

Problem 214. A round steel bar 2 in. in diameter, supported at points 4 ft. apart, deflects .029 in. under a central load of 300 lb. and twists 1.62° in a length of $2\frac{1}{2}$ ft. under a twisting moment of 1500 ft. lb. Find E, G, and Poisson's ratio for the material (see Article 35).

Problem 215. If P and Q denote the unit stresses at the elastic limits of a material in tension and shear respectively, show that when $\frac{P}{Q} < 1$ the material will fail in tension, whereas when $\frac{P}{Q} > 1$ it will fail in shear, when subjected to combined bending and torsion, irrespective of the relative values of the bending and twisting moments.

Solution. Combining Rankine's and Guest's formulas, we have

$$p' - q' = \frac{16\, M_b}{\pi d^3}.$$

Consequently, if the bending moment is zero, $p' = q'$ or $\frac{p'}{q'} = 1$, whereas if it is not zero, $p' > q'$. Similarly, if the twisting moment is zero, $\frac{p'}{q'} = 2$.

Now let F_t and F_s denote the factors of safety in tension and shear respectively. Then

$$\frac{F_t}{F_s} = \frac{\frac{P}{p'}}{\frac{Q}{q'}} = \frac{Pq'}{Qp'}.$$

Since $p' \gtreqless q'$, the fraction $\frac{q'}{p'} \lesseqgtr 1$. Consequently, if $\frac{P}{Q} < 1$ also, then $\frac{F_t}{F_s} < 1$; i.e. $F_t < F_s$ and the material is weaker in tension than in shear. The second part of the theorem is proved in a similar manner.

For a complete discussion of this question see article by A. L. Jenkins, *Engineering News* (London, November 12, 1909), pp. 637–639.

Problem 216. A steel shaft subjected to combined bending and torsion has an elastic limit in tension of 64,600 lb./in.^{2} and an elastic limit in shear of 29,170 lb./in.^{2} Show that Guest's formula applies to this material rather than Rankine's.

Problem 217. A shaft subjected to combined bending and twisting is made of steel for which the elastic limit in tension is 28,800 lb./in.^{2} and the elastic limit in shear is 16,000 lb./in.^{2} Show that if the bending moment is one half the twisting moment, the shaft will be weakest in shear, whereas if the bending moment is twice the twisting moment, it will be weakest in tension.

Problem 218. A closely coiled helical spring is made of $\frac{1}{4}$-in. round steel wire and has 15 coils of mean diameter 3 in. Find its deflection under an axial load of 20 lb., and the stiffness of the spring in pounds per foot of deflection.

Problem 219. A helical-spring transmission dynamometer is made of 20 turns of $\frac{1}{4}$-in. steel wire; mean diameter of coil $2\frac{1}{2}$ in. with 2 turns per inch. Find its axial twist in degrees when transmitting 6 H.P. at 75 R.P.M.

Problem 220. A closely coiled helical spring is made of $\frac{3}{8}$-in. steel wire with coils of 3 in. mean diameter. Find the length of wire required in order that the spring shall deflect $\frac{1}{8}$-in. per pound of load.

CHAPTER VII

SPHERES AND CYLINDERS UNDER UNIFORM PRESSURE

109. Hoop stress. When a hollow sphere or cylinder is subjected to uniform pressure, as in the case of steam boilers, standpipes, gas, water, and steam pipes, fire tubes, etc., the effect of the radial pressure is to produce stress in a circumferential direction, called **hoop stress.** In the case of a cylinder closed at the ends, the pressure on the ends produces longitudinal stress in the side walls in addition to the hoop stress.

If the thickness of a cylinder or sphere is small as compared with its diameter, it is called a **shell.** In analyzing the stress in a thin shell subjected to uniform pressure, such as that due to water, steam, or gas, it may be assumed that the hoop stress is distributed uniformly over any cross section of the shell. This assumption will be made in what follows.

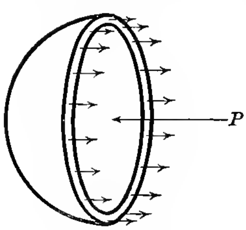


FIG. 108

110. Hoop tension in hollow sphere. Consider a spherical shell subjected to uniform internal pressure, and suppose that the shell is cut into hemispheres by a diametral plane (Fig. 108). Then, if w denotes the pressure per unit of area within the shell, the resultant force acting on either hemisphere is $P = \frac{\pi d^2 w}{4}$, where d is the radius of the sphere. If p denotes the unit tensile stress on the circular cross section of the shell, the total stress on this cross section is πdhp, approximately, where h is the thickness of the shell. Consequently,

$$\frac{\pi d^2 w}{4} = \pi dhp; \text{ whence } p = \frac{wd}{4h},$$

which gives the hoop tension in terms of the radial pressure.

From symmetry, the stress is the same on any diametral cross section. Therefore the equivalent stress at any point of the shell is

$$p_e = \frac{m-1}{m} p = \frac{m-1}{m} \cdot \frac{wd}{4h}.$$

If the value of m is assumed to be $3\frac{1}{3}$, this expression for p_e becomes

$$p_e = .175 \frac{wd}{h}. \tag{63}$$

Problem 221. How great is the stress in a copper sphere 2 ft. in diameter and .25 of an inch thick, under an internal pressure of 175 lb./in.^{2}?

111. Hoop tension in hollow circular cylinder. In the case of a cylindrical shell, its ends hold the cylindrical part together in such a way as to relieve the hoop tension at either extremity. Suppose, then, that the portion of the cylinder considered is so far removed from either end that the influence of the end constraint can be assumed to be zero.

Suppose the cylinder cut in two by a plane through its axis, and consider a section cut out of either half cylinder by two planes perpendicular to the axis, at a distance apart equal to c (Fig. 109). Then the resultant internal pressure P on the strip under consideration is $P = cdw$, and the resultant hoop tension is $2\,chp$, where the letters have the same meaning as in the preceding article. Consequently, $cdw = 2\,chp$; whence

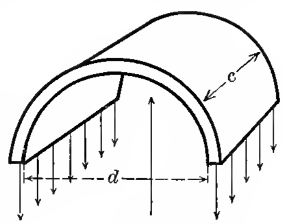


FIG. 109

$$p = \frac{dw}{2h}. \tag{64}$$

If the longitudinal stress is zero, $p_e = p$.

This result is applicable to shells under both inner and outer pressure, if P is taken to be the excess of the internal over the external pressure.

Problem 222. A cast-iron water pipe is 24 in. in diameter and 2 in. thick. What is the greatest internal pressure which it can withstand?

112. Longitudinal stress in hollow circular cylinder. If the ends of a cylinder are fastened to the cylindrical part, the internal pressure against the ends produces longitudinal stresses in the side walls. In this case the cylindrical part is subjected both to hoop tension and to longitudinal tension.

To find the amount of the longitudinal tension, consider a cross section of the cylinder near its center, where the influence of the end restraints can be assumed to be zero (Fig. 110). Then the resultant pressure on either end is $P = \frac{\pi d^2 w}{4}$, and the resultant longitudinal stress on the cross section is πdhp. Therefore

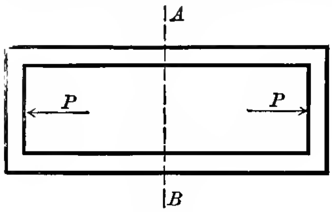


FIG. 110

$$\frac{\pi d^2 w}{4} = \pi dhp; \text{ whence } p = \frac{wd}{4h}.$$

This is the same formula as for the sphere, which was to be expected, since the cross section is the same in both cases.

If p_l denotes the longitudinal stress and p_h the hoop tension, then $p_l = \frac{wd}{4h}$, $p_h = \frac{wd}{2h}$; and, consequently, the equivalent stress p_e is

$$p_e = p_h - \frac{1}{m} p_l = \frac{2m-1}{4m} \cdot \frac{wd}{h}.$$

If $m = 3\frac{1}{3}$, this becomes

$$\textbf{(65)} \qquad \boldsymbol{p_e = .425 \frac{wd}{h}}.$$

Formula (**65**) is the one to be used in finding the tensile stress in a thin cylinder subjected to uniform internal pressure, in which the ends are held together by the body of the cylinder and not by independent stays or fixed supports.

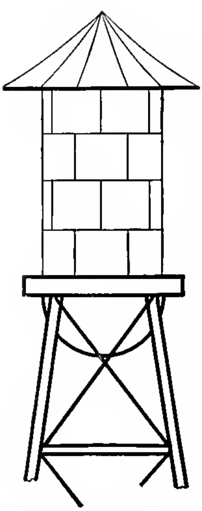


FIG. 111

Problem 223. An elevated water tank is cylindrical in form with a hemispherical bottom (Fig. 111). The diameter of the tank is 20 ft. and its height 52 ft., exclusive of the bottom. If the tank is to be built of wrought iron and the factor of safety is taken to be 6, what should be the thickness of the bottom plates, and also of those in the body of the tank near its bottom?

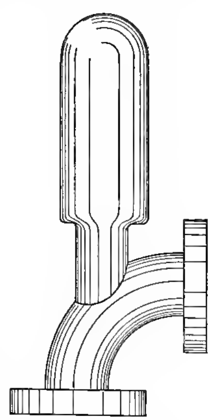


FIG. 112

NOTE. Formulas (**63**) and (**65**) give the required thickness of the plates, provided the tank is without joints. The bearing power of the rivets at the joints, however, is, in general, the consideration which determines the thickness of the plates (Art. 122).

Problem 224. A marine boiler shell is 16 ft. long, 8 ft. in diameter, and 1 in. thick. What is the stress in the shell for a working gauge pressure of 160 lb./in.^{2}?

Problem 225. The air chamber of a pump is made of cast iron of the form shown in Fig. 112. If the diameter of the air chamber is 10 in. and its height 24 in., how thick must the walls of the air chamber be made to stand a pressure of 500 lb./in.^{2} with a factor of safety of 4?

*** 113. Differential equation of elastic curve for circular cylinder.** A cylindrical shell subjected to internal pressure is in a condition of stable equilibrium, for the internal pressure tends to preserve the cylindrical form of the shell, or to restore it to this form if, by any cause, the cylinder is flattened or otherwise deformed. A cylindrical shell which is subjected to external pressure, however, is in a condition of unstable equilibrium, for any deviation from a cylindrical form tends to be increased rather than diminished by the stress. In this respect thin hollow cylinders under external pressure are in a state of strain similar to that in a column, and the method of finding the critical pressure just preceding collapse is similar to that for finding the critical load for a column, as explained in the derivation of Euler's formula.

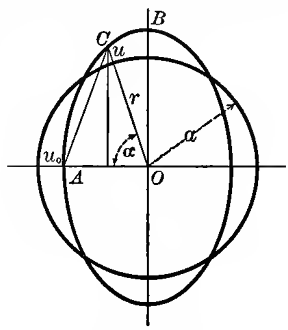


FIG. 113

Consider a thin hollow cylinder which is subjected to a uniform external pressure of amount w per unit of area, and suppose that in some way the cylinder has been compressed in one direction so that it assumes the flattened form shown in Fig. 113. The first step in the solution of the problem is to find the differential equation of the elastic curve in curvilinear coördinates, or, in other words, the differential equation of the elastic curve of the flattened cylinder referred to its original circular form.

In polar coördinates let O be the origin and OA the initial line. Also, let a denote the radius of the circular cylinder, and r the radius vector of the flattened or elliptical form. Now suppose that the circular wall of the cylinder is considered as a piece which was originally straight and has been made to assume a circular form by a

* For a brief course the remainder of this chapter may be omitted.

bending moment M' Then, if ρ denotes the radius of curvature, from Article 66,

$$\frac{1}{\rho} = \frac{M'}{EI}.$$

Again, suppose that this circular cylinder is made to assume the flattened form as the result of an additional bending moment M, and let ρ' denote the corresponding radius of curvature. Then

$$\frac{1}{\rho'} = \frac{M' + M}{EI}.$$

Consequently,

(66) $$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{M}{EI}.$$

From the differential calculus,

$$\rho' = \frac{\left[r^2 + \left(\frac{dr}{da}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{da}\right)^2 - r\frac{d^2r}{da^2}}.$$

If the deformation is small, $\frac{dr}{da}$ is infinitesimal, and r differs infinitesimally from a. Therefore, neglecting infinitesimals of an order higher than the second, the expression for ρ' becomes

$$\rho' = \frac{a^3}{a^2 - a\frac{d^2r}{da^2}} = \frac{a^2}{a - \frac{d^2r}{da^2}},$$

and, consequently,

$$\frac{1}{\rho'} = \frac{1}{a} - \frac{1}{a^2}\cdot\frac{d^2r}{da^2}.$$

Since $\rho = a$, $\frac{1}{\rho} = \frac{1}{a}$, and therefore

(67) $$\frac{1}{\rho'} - \frac{1}{\rho} = -\frac{1}{a^2}\cdot\frac{d^2r}{da^2}.$$

Comparing equations (66) and (67),

(68) $$\frac{1}{a^2}\cdot\frac{d^2r}{da^2} = \pm\frac{M}{EI}.*$$

*The sign $\pm$ is used because the calculus expression for ρ' contains a square root in the numerator.

Now let u denote the distance between the circle and the ellipse measured radially. Then

$$r = a \pm u,$$

or, if u is assumed to be positive when it lies outside the circle and negative when it lies inside,

$$r = u + a.$$

Differentiating both sides of this equation with respect to α,

$$\frac{dr}{d\alpha} = \frac{du}{d\alpha}, \qquad \frac{d^2r}{d\alpha^2} = \frac{d^2u}{d\alpha^2}.$$

Also, if dl is the length of an infinitesimal arc of the circle, $ad\alpha = dl$. Substituting these values in equation (**68**), it becomes

$$\text{(69)} \qquad EI\frac{d^2u}{dl^2} = \pm M,$$

which is the required differential equation of the elastic curve in the curvilinear coördinates l and u.

114. Crushing strength of hollow circular cylinder. As a continuation of the preceding article, let it be required to find the external pressure which is just sufficient to cause the cylinder to retain its flattened form, or, in other words, the critical external pressure just preceding collapse.

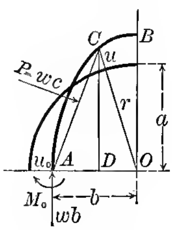


FIG. 114

In Fig. 114 let OA and OB be axes of symmetry; then it is sufficient to consider merely the quadrant AOB. Let c denote the length of the chord AC, and let w be the unit external pressure. Then for a section of the cylinder of unit length the external pressure P on the curved strip AC is

$$P = wc.$$

Now let M_0 denote the bending moment at the point A. The tangential force at this point is equal to the resultant pressure on OA, or wb. Consequently the bending moment M at the point C is

$$M = M_0 + wb \cdot AD - wc \cdot \frac{c}{2} = M_0 + w\left(b \cdot AD - \frac{c^2}{2}\right).$$

In the triangle OAC,

$$\overline{OC}^2 = \overline{AC}^2 + \overline{AO}^2 - 2\,AO\cdot AD, \text{ or}$$

$$r^2 = c^2 + b^2 - 2\cdot b\cdot AD,$$

from which

$$b\cdot AD - \frac{c^2}{2} = \frac{b^2 - r^2}{2}.$$

Hence

$$M = M_0 + \frac{w(b^2 - r^2)}{2}.$$

Since $r = u + a$ and $a = b - u_0$,*

$$M = M_0 + \frac{w}{2}(a^2 + 2\,au_0 + u_0^2 - a^2 - 2\,au - u^2)$$

$$= M_0 + \frac{w}{2}(u_0 - u)(u_0 + u + 2\,a).$$

Since u and u_0 are both infinitesimal, $u_0 + u$ (or the difference between the absolute values of u and u_0) is negligible in comparison with $2\,a$. Therefore

$$M = M_0 - wa(u - u_0),$$

and, consequently, the differential equation of the elastic curve becomes

$$EI\frac{d^2u}{dl^2} = M_0 - wa(u - u_0).$$

The general integral of this differential equation is found to be

$$(70)\quad u = u_0 + \frac{M_0}{wa} + C_1 \sin\sqrt{\frac{wa}{EI}}\,l + C_2 \cos\sqrt{\frac{wa}{EI}}\,l,$$

in which C_1 and C_2 are the undetermined constants of integration.† This may be verified by substituting the integral in the above differential equation.

To determine C_1 and C_2 it is only necessary to make use of the terminal conditions at A and B. At the point A, $l = 0$, $\frac{du}{dl} = 0$, and $u = u_0$. Substituting these values in equation (**70**) and its first derivative, it is found that

$$C_1 = 0 \quad\text{and}\quad C_2 = -\frac{M_0}{wa}.$$

* Throughout this discussion it should be borne in mind that u_0 is a negative quantity.

† See Johnson, *Treatise on Ordinary and Partial Differential Equations*, 3d ed., pp. 85–86.

Hence the integral becomes

$$u = \frac{M_0}{wa} + u_0 - \frac{M_0}{wa}\cos\sqrt{\frac{wa}{EI}}\,l,$$

or

$$(71) \qquad u = \frac{M_0}{wa}\left(1 - \cos\sqrt{\frac{wa}{EI}}\,l\right) + u_0.$$

At the upper end of the quadrant B the conditions are $l = \frac{\pi a}{2}$ and $\frac{du}{dl} = 0$. Substituting these values in the first differential coefficient obtained from equation (71), namely,

$$\frac{du}{dl} = \frac{M_0}{wa}\sqrt{\frac{wa}{EI}}\sin\sqrt{\frac{wa}{EI}}\,l,$$

we have

$$\sin\left(\sqrt{\frac{wa}{EI}}\cdot\frac{\pi a}{2}\right) = 0;$$

whence

$$\sqrt{\frac{wa}{EI}}\cdot\frac{\pi a}{2} = \lambda\pi,$$

where λ is an arbitrary integer. Choosing the smallest value of λ, namely 1, this condition becomes

$$\sqrt{\frac{wa}{EI}}\cdot\frac{a}{2} = 1;$$

whence

$$(72) \qquad w = \frac{4EI}{a^3}.$$

If the thickness of the tube is denoted by h, then, for a section of unit length, $I = \frac{h^3}{12}$, and formula (72) becomes

$$(73) \qquad w = \frac{E}{3}\left(\frac{h}{a}\right)^3.$$

Formula (73) gives the critical pressure just preceding collapse; that is to say, it gives the maximum external pressure w per unit of area which a cylindrical tube of thickness h can stand without crushing.

Problem 226. What is the maximum external pressure which a cast-iron pipe 18 in. in diameter and $\frac{1}{2}$ in. thick can stand without crushing?

Problem 227. In a fire-tube boiler the tubes are of drawn steel, 2 in. internal diameter and $\frac{1}{8}$ in. thick. What is the factor of safety for a working gauge pressure of 200 lb./in.^{2}?

115. Thick cylinders; Lamé's formulas. Consider a thick circular cylinder of external radius a and internal radius b, which is subjected to the action of either internal or external uniform pressure, or to both. Suppose a section is cut out of the cylinder by two planes perpendicular to the axis at a unit's distance apart, and consider a small sector $ABCD$ of angle α cut out of the ring so obtained, as shown in Fig. 115. Let p_h denote the tangential stress, or hoop stress, acting on this infinitesimal element, p_r the radial stress acting on the inner surface AD, and $p_r + dp_r$ the radial stress acting on the outer surface BC.

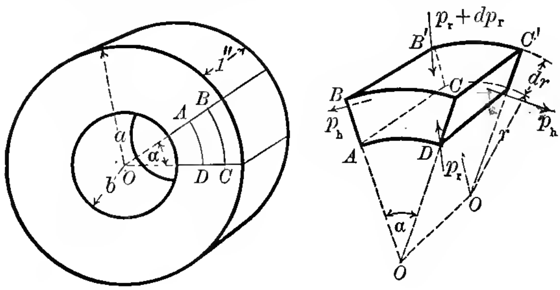


Fig. 115

Then the internal and external radii being r and $r + dr$ respectively, the length of AD is $r\alpha$ and of BC is $(r + dr)\alpha$. Since the width of the piece is unity, the resultant radial force acting on the piece, or the difference between the pressure on the inner and outer surfaces, is $(p_r + dp_r)(r + dr)\alpha - p_r r\alpha$. Therefore, since the resultant of the hoop stress in a radial direction is $(p_h\alpha)\,dr$, in order that the radial stresses shall equilibrate,

$$(p_r + dp_r)(r + dr)\alpha - p_r r\alpha = p_h \alpha dr;$$

or, neglecting infinitesimals of an order higher than the second,

$$p_r dr + r dp_r = p_h dr;$$

which may be written

$$\text{(74)} \qquad \frac{d}{dr}(rp_r) = p_h.$$

If the ends of the cylinder are free from restraint, or if the cylinder is subjected to a *uniform* longitudinal stress, the longitudinal deformation must be constant throughout the cylinder. The longitudinal deformation, however, is due to the lateral action of p_r and p_h, and is of amount $\frac{p_r}{mE} + \frac{p_h}{mE}$, or $\frac{1}{mE}(p_r + p_h)$, in which m denotes Poisson's constant. Therefore, if this expression is constant, $p_r + p_h$ must be constant, and hence

$$p_r + p_h = k,$$

where k is a constant. Consequently, $p_h = k - p_r$, and substituting this value of p_h in equation (**74**) and multiplying by r, it becomes

$$krdr = 2\,rp_rdr + r^2dp_r,$$

which may be written

$$kr = \frac{d}{dr}(r^2p_r).$$

Integrating,

$$r^2p_r = \frac{kr^2}{2} + C_1,$$

in which C_1 is the constant of integration; whence

$$p_r = \frac{k}{2} + \frac{C_1}{r^2}. \tag{75}$$

Also, since $p_h = k - p_r$,

$$p_h = \frac{k}{2} - \frac{C_1}{r^2}. \tag{76}$$

Now suppose that the cylinder is subjected to a uniform internal pressure of amount w_i per unit of area, and also to a uniform external pressure of amount w_e per unit of area. Then $p_r = w_e$ when $r = a$, and $p_r = w_i$ when $r = b$. Substituting these values in equation (**75**),

$$w_e = \frac{k}{2} + \frac{C_1}{a^2}, \qquad w_i = \frac{k}{2} + \frac{C_1}{b^2};$$

whence

$$C_1 = \frac{a^2b^2(w_e - w_i)}{b^2 - a^2}, \qquad k = \frac{2\,(w_ea^2 - w_ib^2)}{a^2 - b^2}.$$

Therefore, substituting these values of C_1 and k in equations (**75**) and (**76**), they become

$$\begin{cases} \boldsymbol{p_r = \dfrac{w_ea^2 - w_ib^2}{a^2 - b^2} - \dfrac{a^2b^2(w_e - w_i)}{(a^2 - b^2)r^2},} \\[2ex] \boldsymbol{p_h = \dfrac{w_ea^2 - w_ib^2}{a^2 - b^2} + \dfrac{a^2b^2(w_e - w_i)}{(a^2 - b^2)r^2},} \end{cases} \tag{77}$$

which give the radial and hoop stresses in a thick cylinder subjected to internal and external pressure. Equations (**77**) are known as **Lamé's formulas.**

116. Maximum stress in thick cylinder under uniform internal pressure. Consider a thick circular cylinder which is subjected only to internal pressure. Then $w_e = 0$, and equations (**77**) become

$$(78) \qquad p_r = \frac{w_i b^2}{a^2 - b^2}\left(\frac{a^2}{r^2} - 1\right), \quad p_h = -\frac{w_i b^2}{a^2 - b^2}\left(\frac{a^2}{r^2} + 1\right).$$

Since p_h is negative, the hoop stress in this case is tension.

Since p_r and p_h both increase as r decreases, the maximum stress occurs on the inner surface of the cylinder, where

$$r = b, \quad p_r = w_i, \quad \text{and} \quad p_h = -\frac{w_i(a^2 + b^2)}{a^2 - b^2}.$$

Clearing the latter of fractions, we have $\frac{a^2}{b^2} = \frac{p_h + w_i}{p_h - w_i}$, whence the thickness of the tube, $h = a - b$, is given by

$$(79) \qquad h = b\left(\sqrt{\frac{p_h + w_i}{p_h - w_i}} - 1\right).$$

Moreover, the equivalent stress for a point on the inner surface of the cylinder is

$$p_e = p_h - \frac{1}{m}p_r = -\frac{w_i}{m(a^2 - b^2)}[(m-1)b^2 + (m+1)a^2].$$

If $m = 3\frac{1}{3}$, the absolute value of the equivalent stress becomes

$$(80) \qquad p_e = \frac{w_i}{a^2 - b^2}(.7\,b^2 + 1.3\,a^2).$$

This may also be written

$$p_e = w\frac{.7b^2 + 1.3a^2}{a^2 - b^2} = w\frac{a^2 + b^2}{a^2 - b^2} + w\frac{.3\,a^2 - .3\,b^2}{a^2 - b^2} = p_h + .3\,w.$$

Problem 228. Find the thickness necessary to give to a steel locomotive cylinder of 22 in. internal diameter, if it is required to withstand a maximum steam pressure of 150 lb./in.^{2} with a factor of safety of 10.

Problem 229. In a four-cycle gas engine the cylinder is of steel with an internal diameter of 6 in., and the initial internal pressure is 200 lb./in.^{2} absolute. With a factor of safety of 15, how thick should the walls of the cylinder be made?

Problem 230. The steel cylinder of an hydraulic press has an internal diameter of 5 in. and an external diameter of 7 in. With a factor of safety of 3, how great an internal pressure can the cylinder withstand?

117. Bursting pressure for thick cylinder. Let u_t denote the ultimate tensile strength of the material of which the cylinder is composed. Then, from equation (**79**), the maximum allowable internal pressure w_i is obtained from the equation

$$u_t = -\frac{w_i}{m(a^2 - b^2)}[(m-1)b^2 + (m+1)a^2];$$

whence

$$w_i = \frac{u_t m(a^2 - b^2)}{(m-1)b^2 + (m+1)a^2}. \tag{81}$$

If m is assumed to be $3\frac{1}{3}$, this formula becomes

$$\boldsymbol{w_i = \frac{u_t(a^2 - b^2)}{.7\,b^2 + 1.3\,a^2}}. \tag{82}$$

Equations (**81**) and (**82**) give the maximum internal pressure w_i which the cylinder can stand without bursting.

Problem 231. A wrought-iron pipe is 4 in. in external diameter and .25 in. thick. What head of water will it stand without bursting?

Problem 232. Under a water head of 200 ft., what is the factor of safety in the preceding problem?

118. Maximum stress in thick cylinder under uniform external pressure. Consider a thick circular cylinder subjected only to external pressure. In this case $w_i = 0$ and equations (**77**) become

$$p_r = \frac{w_e a^2}{a^2 - b^2}\left(1 - \frac{b^2}{r^2}\right),$$

$$p_h = \frac{w_e a^2}{a^2 - b^2}\left(1 + \frac{b^2}{r^2}\right).$$

Since p_h is positive, the hoop stress in this case is compression.

For a point on the inner surface of the cylinder

$$r = b, \quad p_r = 0, \quad \text{and} \quad p_h = \frac{2\,w_e a^2}{a^2 - b^2}.$$

Since the radial stress is zero on the inner surface, the equivalent stress is equal to the hoop stress, that is,

$$\boldsymbol{p_e = \frac{2\,w_e a^2}{a^2 - b^2}}.$$

Problem 233. A wrought-iron cylinder is 8 in. in external diameter and $1\frac{1}{2}$ in. thick. How great an external pressure can it withstand?

119. Thick cylinders built up of concentric tubes. From equations (77), it is evident that in a thick cylinder subjected to internal pressure the stress is greatest on the inside of the cylinder, and decreases toward the outside. In order to equalize the stress throughout the cylinder and thus obtain a more economical use of material, the device is resorted to of forming the cylinder of several concentric tubes and producing an initial compressive stress on the inner ones. For instance, in constructing the barrel of a cannon, or the cylinder of an hydraulic press, the cylinder is built up of two or more tubes. The outer tubes in this case are made of somewhat smaller diameter than the inner tubes, and then each is heated until it has expanded sufficiently to be slipped over the one next smaller. In cooling, the metal of the outer tube contracts, thus producing a compressive stress in the inner tube and a tensile stress in the outer tube. If, then, this composite tube is subjected to internal pressure, the first effect of the hoop tension thus produced is to relieve the initial compressive stress in the inner tube and increase that in the outer tube. Thus the resultant stress in the inner tube is equal to the difference between the initial stress and that due to the internal pressure, whereas the resultant stress in the outer tube is the sum of these two. In this way the strain is distributed more equally throughout the cylinder. It is evident that the greater the number of tubes used in building up the cylinder, the more nearly can the strain be equalized.

The preceding discussion of the stress in thick tubes can also be applied to the calculation of the stress in a rotating disk. For example, a grindstone is strained in precisely the same way as a thick tube under internal pressure, the load in this case being due to centrifugal force instead of to the pressure of a fluid or gas.

120. Practical formulas for the collapse of tubes under external pressure. A more rigorous analysis of the stress in thin tubes, due to external pressure, than that given in Article 114, using Poisson's ratio $\frac{1}{m}$ of transverse to longitudinal deformation, gives the formula *

$$w = \frac{E}{4\left(1 - \frac{1}{m^2}\right)} \left(\frac{h}{a}\right)^3,$$

* Love, *Math. Theory of Elast.*, Vol. II, pp. 308–316.

or, in terms of the diameter $D = 2a$,

$$w = \frac{2E}{1 - \frac{1}{m^2}} \left(\frac{h}{D}\right)^3.$$

This formula, however, is based on the assumptions that the tube is perfectly symmetrical, of uniform thickness, and of homogeneous material, — conditions which are never fully realized in commercial tubes. From recent experiments on the collapse of tubes,* however, it is now possible to determine the practical limitations of this formula, and so modify it, by a method similar to that by which the Gordon-Rankine column formula was deduced from Euler's formula (Articles 88, 89), as to obtain a rational formula which shall nevertheless conform closely to experimental results. By determining the ellipticity, or deviation from roundness, and the variation in thickness of the various types of tubes covered by the tests mentioned above, it is found that by introducing empirical constants the rational formulas can be made to fit experimental results as closely as any empirical formulas, with the advantage of being unlimited in their range of application.† The formula so obtained is

$$\boldsymbol{w = \frac{2EC}{1 - \frac{1}{m^2}} \left(\frac{h}{D}\right)^3}, \qquad \left\{\begin{array}{l}\text{for thin tubes} \\ \dfrac{h}{D} \leqq .023\end{array}\right.$$

where $h =$ average thickness of tube in inches,

$D =$ maximum outside diameter in inches,

$\frac{1}{m} =$ Poisson's ratio $= .3$ for steel,

$C = .69$ for lap-welded steel boiler flues,

$= .76$ for cold-drawn seamless steel flues,

$= .78$ for drawn seamless brass tubes.

By a similar procedure for thick tubes $\left(\frac{h}{D} > .023\right)$ a practical

* Carman, "Resist. of Tubes to Collapse," *Univ. Ill. Bull.*, Vol. III, No. 17; Stewart, "Collap. Press. Lap-Welded Steel Tubes," *Trans. A.S.M.E.*, 1906, pp. 730–820.

† Slocum, "The Collapse of Tubes under External Pressure," *Engineering.* London, January 8, 1909. Also abstract of same article in Kent, 8th ed., 1910, pp. 320–322.

rational formula has been obtained from Lamé's formula, Article 118, for this case also, namely

$$w = \frac{2\,Khu_c}{D}\left(1 - K\frac{h}{D}\right), \qquad \left\{\begin{array}{l}\text{for thick tubes}\\ \dfrac{h}{D} > .023\end{array}\right.$$

where u_c = ultimate compressive strength of the material,
$K = .89$ for lap-welded steel boiler flues.

Only one value of K is given, as the experiments cited were all made on one type of tube.

The correction constants C and K include corrections both for ellipticity, or flattening of the tube, and for variation in thickness. Thus if the correction for ellipticity is denoted by C_1 and the correction for variation in thickness by C_2, we have

$$C_1 = \frac{\text{Minimum outside diameter}}{\text{Maximum outside diameter}},$$

$$C_2 = \frac{\text{Minimum thickness}}{\text{Average thickness}},$$

and the correction constants C and K are therefore defined as

$$C = C_1^3\,C_2^3,$$
$$K = C_1 C_2.$$

By an experimental determination of C_1 and C_2 the formulas can therefore be applied to any given type of tube.

121. Shrinkage and forced fits. In machine construction shrinkage and forced or pressed fits are frequently employed for connecting certain parts, such as crank disk and shaft, wheel and axle, etc. To make such a connection the shaft is finished slightly larger than the hole in the disk or ring in which it belongs. The shaft is then either tapered slightly at the end and pressed into the ring cold, or the ring is enlarged by heating until it will slip over the shaft, in which case the shrinkage due to cooling causes it to grip the shaft.

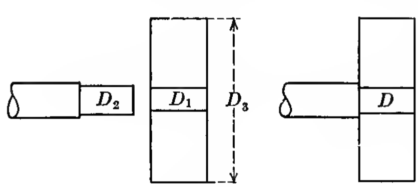


Fig. 116

To analyze the stresses arising from shrinkage and forced fits, let D_1 denote the diameter of the hole in the ring or disk, and D_2 the diameter of the shaft (Fig. 116). When shrunk or forced together, D_1 must increase slightly and D_2 decrease slightly, i.e. D_1 and D_2 must of necessity take the same value D. Consequently the circumference of the hole changes from πD_1 to πD, and hence the unit deformation s_1 of a fiber on the inner surface of the hole is

$$s_1 = \frac{\pi D - \pi D_1}{\pi D_1} = \frac{D - D_1}{D_1}.$$

Similarly the unit deformation s_2 of a fiber on the surface of the shaft is

$$s_2 = \frac{\pi D_2 - \pi D}{\pi D_2} = \frac{D_2 - D}{D_2}.$$

From Hooke's law, $\frac{p}{s} = E$, we have therefore for the unit stress p_1 on the inside of the disk

$$\frac{p_1}{E_1} = s_1 = \frac{D - D_1}{D_1};$$

and for the unit stress p_2 on the surface of the shaft

$$\frac{p_2}{E_2} = s_2 = \frac{D_2 - D}{D_2}.$$

Adding these two equations to eliminate the unknown quantity D, the result is

$$\frac{p_1 D_1}{E_1} + \frac{p_2 D_2}{E_2} = D_2 - D_1 = K,$$

where K denotes the **allowance**, or difference in diameter of shaft and hole. For a thick disk or heavy ring this allowance K may be determined from the nominal diameter D of the shaft by means of the following empirical formulas.*

For shrinkage fits, $$K = \frac{\frac{17}{16} D + \frac{1}{2}}{1000},$$

For pressed fits, $$K = \frac{2 D + \frac{1}{2}}{1000},$$

For driven fits, $$K = \frac{\frac{1}{2} D + \frac{1}{2}}{1000}.$$

* S. H. Moore, *Trans. Am. Soc. Mech. Eng.*, Vol. XXIV.

For thin rings, however, the allowance given by these formulas will be found to produce stresses in the ring entirely too large for safety. In deciding on the allowance for any given class of work, the working stresses in shaft and ring may first be assigned and the allowance then determined from the formulas given below so that the actual stresses shall not exceed these values.

From Lamé's formulas the stresses p_1 and p_2 may be obtained in terms of the unit pressure between the surfaces in contact. Thus from formula (**80**) the equivalent stress on the inside of the hole is

$$p_1 = p_e = \frac{w}{D_3^2 - D_1^2}(.7\,D_1^2 + 1.3\,D_3^2),$$

where D_3 denotes the outside diameter of the ring, while, by substituting $r = a$ and $b = 0$ in the equations of Article 118, the stresses on the outer surface of the shaft are found to be

$$p_h = w, \qquad p_r = w,$$

and consequently

$$p_2 = p_h - \frac{1}{m}p_r = .7\,w.$$

Eliminating w between these expressions for p_1 and p_2, we have

$$p_1 = \frac{p_2}{.7}\left(\frac{.7\,D_1^2 + 1.3\,D_3^2}{D_3^2 - D_1^2}\right).$$

Now to simplify the solution, let the coefficient of p_2 be denoted by H; that is, let

$$H = \frac{.7\,D_1^2 + 1.3\,D_3^2}{.7\,(D_3^2 - D_1^2)},$$

in which case

$$p_1 = Hp_2.$$

Eliminating p_1 between this relation and the above expression for the allowance K, we have finally

$$\boldsymbol{p_2 = \frac{K}{\dfrac{HD_1}{E_1} + \dfrac{D_2}{E_2}},}$$

$$\boldsymbol{p_1 = Hp_2.}$$

In applying these formulas the constant H is first computed from the given dimensions of the parts. If the allowance K is given, the unit

stresses p_1 and p_2 in ring and shaft are then found from the above. If K is to be determined, a safe value for the stress in the ring, p_1, is assigned, and p_2 calculated from the second equation. This value is then substituted in the first equation and K calculated.

The following problem illustrates the application of the formulas.

Problem 234. A cast-iron gear, 8 in. external diameter, 3 in. wide, and $1\frac{3}{4}$ in. internal diameter, is to be forced on a steel shaft. Find the stresses developed, the pressure required to force the gear on the shaft, and the tangential thrust required to shear the fit, i.e. produce relative motion between gear and shaft.

Solution. From the formula $K = \dfrac{2D + \frac{1}{2}}{1000}$ the allowance is found to be .004 in., making the diameter of the shaft $D_2 = 1.754$ in. Also since $D_1 = 1.75$ in., $D_3 = 8$ in., we have $H = 2.0007$. Hence assuming $E_1 = 15{,}000{,}000$ lb./in.^{2} and $E_2 = 30{,}000{,}000$ lb./in.^{2}, we have

$$p_1 = 13{,}713 \text{ lb./in.}^2, \qquad p_2 = 27{,}436 \text{ lb./in.}^2$$

To find the pressure required to force the gear on the shaft it is first necessary to calculate the pressure between the surfaces in contact. From the relation $p_2 = .7\,w$ this amounts to

$$w = 39{,}194 \text{ lb./in.}^2$$

The coefficient of friction depends on the nature of the surfaces in contact. Assuming it to be $\mu = .15$ as an average value, and with a nominal area of contact of $\pi \times 1\frac{3}{4} \times 3 = 16.485$ in.^{2}, the total pressure P required is

$$P = 16.485 \times 39{,}194 \times .15 = 96{,}917 \text{ lb.} = 48.5 \text{ tons.}$$

To find the torsional resistance of the fit, we have, as above

$$\text{Bearing area} = 16.485 \text{ in.}^2, \qquad \text{Unit pressure} = 39{,}194 \text{ lb./in.}^2,$$

$$\mu = .15, \text{ radius of shaft} = .875 \text{ in.}$$

Hence the torsional resistance is

$$M_t = 16.485 \times 39{,}194 \times .15 \times .875 = 85{,}000 \text{ in. lb.}$$

Consequently the tangential thrust on the teeth of the gear necessary to shear the fit is

$$\tfrac{85000}{4} = 21{,}250 \text{ lb.} = 10.6 \text{ tons.}$$

122. Riveted joints. In structural work such as plate girders, trusses, etc., and also in steam boilers, standpipes, and similar constructions, the connections between the various members are made by riveting the parts together. As the holes for the rivets weaken the members so joined, the strength of the structure is determined by the strength of the joint.

Failure of a riveted joint may occur in various ways, namely, by shearing across the rivet, by crushing the rivet, by crushing the plate in

front of the rivet, by shearing the plate, i.e. pulling out the rivets, or by tearing the plate along the line of rivet holes. Experience has shown, however, that failure usually occurs either by shearing across the rivet or by tearing the plate along the line of rivet holes.

The strength of any given type of riveted joint is expressed by what is called its **efficiency**, defined as

$$\textbf{Efficiency of riveted joint} = \frac{\textbf{strength of joint}}{\textbf{strength of unriveted member}}.$$

Thus if d (Fig. 117), denotes the diameter of a rivet and c the distance between rivet holes, or **pitch** of the rivets as it is called, the efficiency of the joint against tearing of the plate along the line of rivets is

$$e = \frac{c - d}{c}.$$

To determine the efficiency of the joint against shearing across the rivets, let q denote the ultimate shearing strength of the rivet and p the ultimate tensile strength of the plate. Then for a single lap joint (Fig. 117), if h denotes the thickness of the plate, the area corresponding to one rivet is hd, and the area in shear for each rivet is $\frac{\pi d^2}{4}$; consequently the efficiency of this type of joint against rivet shearing is

$$e = \frac{\pi d^2 q}{4\,chp}.$$

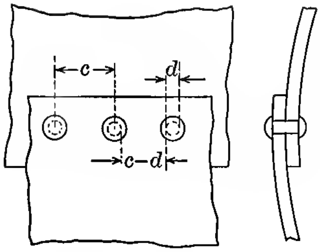
For an economical design these two efficiencies should be equal. For practical reasons, however, it is not generally possible to make these exactly equal, and in this case the smaller of the two determines the strength of the joint.

For a double-riveted lap joint the efficiency against tearing of the plate is

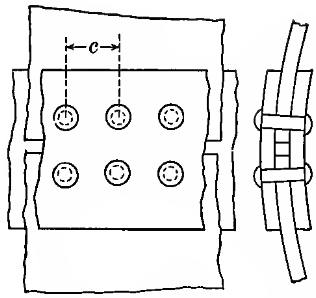
$$e = \frac{c - d}{c},$$

as above; but since in this case there are two rivets for each strip of length c, the efficiency against rivet shear is

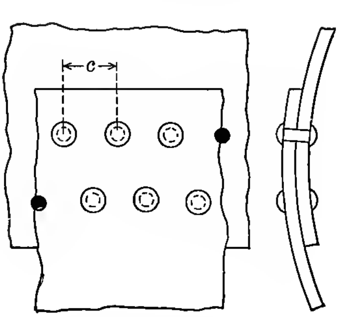
$$e = \frac{\pi d^2 q}{2\,chp}.$$



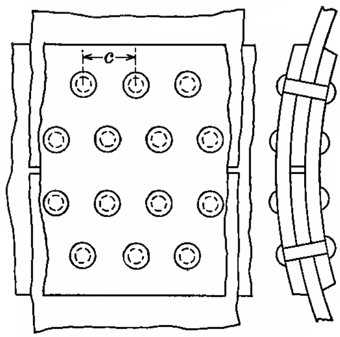
SINGLE-RIVETED LAP JOINT
EFFICIENCY 50–60 PER CENT



SINGLE-RIVETED BUTT JOINT
EFFICIENCY 76–78 PER CENT



DOUBLE-RIVETED LAP JOINT
EFFICIENCY 70–72 PER CENT



DOUBLE-RIVETED BUTT JOINT
EFFICIENCY 82–83 PER CENT

FIG. 117

Similarly for a single-riveted butt joint with two cover plates the efficiency of the joint against tearing of the plate is

$$e = \frac{c - d}{c},$$

and against rivet shear is

$$e = \frac{\pi d^2 q}{2\,chp}.$$

For a double-riveted butt joint with two cover plates the efficiency against tearing of the plate is

$$e = \frac{c - d}{c},$$

and against rivet shear is

$$e = \frac{\pi d^2 q}{chp}.$$

The average efficiencies of various types of riveted joints as used in steam boilers are given in Fig. 117.

In designing steam-boiler shells it is customary in this country to determine first the thickness of shell plates by the following rule.

To find the thickness of shell plates, multiply the maximum steam pressure to be carried (safe working pressure in lb./in.^{2}) by half the diameter of the boiler in inches. This gives the hoop stress in the shell per unit of length. Divide this result by the safe working stress (working stress = ultimate strength, usually about 60,000 lb./in.^{2}, divided by the factor of safety, say 4 or 5) and divide the quotient by the average efficiency of the style of joint to be used, expressed as a decimal. The result will be the thickness of the shell plates expressed in decimal fractions of an inch.

Having determined the thickness of shell plates by this method, the diameter of the rivets is next found from the empirical formula

$$d = k\sqrt{t},$$

where $k = 1.5$ for lap joints and $k = 1.3$ for butt joints with two cover plates.

The pitch of the rivets is next determined by equating the strength of the plate along a section through the rivet holes to the strength of the rivets in shear, and solving the resulting equation for c.

To illustrate the application of these rules, let it be required to design a boiler shell 48 in. in diameter to carry a steam pressure of 125 lb./in.^{2} with a double-riveted, double-strapped butt joint.

By the above rule for thickness of shell plates we have

$$h = \frac{125 \times \frac{48}{2}}{\frac{60,000}{5} \times .82} = .3, \text{ say } \tfrac{5}{16} \text{ in.}$$

The diameter of rivets is then

$$d = 1.3\sqrt{\tfrac{5}{16}} = .73, \text{ say } \tfrac{3}{4} \text{ in.}$$

To determine the pitch of the rivets, the strength of the plate for a section of width c on a line through the rivet holes is

$$(c - d)hp = (c - \tfrac{3}{4})\tfrac{5}{16} \times 60,000,$$

and the strength of the rivets in shear for a strip of this width is

$$4 \times \frac{\pi d^2}{4} q = \pi \frac{9}{16} \times 40{,}000.$$

Equating these two results and solving for c, we have

$$(c - \tfrac{3}{4})\tfrac{5}{16} \times 60{,}000 = \pi \tfrac{9}{16} \times 40{,}000,$$

whence

$$c = 4.5 \text{ in.}$$

As a check on the correctness of our assumptions the efficiency of the joint is found to be

$$e = \frac{c - d}{c} = \frac{4.5 - .75}{4.5} = .83.$$

For bridge and structural work the following empirical rules are representative of American practice.*

The pitch or distance from center to center of rivets should not be less than 3 diameters of the rivet. In bridge work the pitch should not exceed 6 inches, or 16 times the thickness of the thinnest outside plates except in special cases hereafter noted. In the flanges of beams and girders, where plates more than 12 inches wide are used, an extra line of rivets with a pitch not greater than 9 inches should be driven along each edge to draw the plates together.

At the ends of compression members the pitch should not exceed 4 diameters of the rivet for a length equal to twice the width or diameter of the member.

In the flanges of girders and chords carrying floors, the pitch should not exceed 4 inches.

For plates in compression the pitch in the direction of the line of stress should not exceed 16 times the thickness of the plate, and the pitch in a direction at right angles to the line of stress should not exceed 32 times the thickness, except for cover plates of top chords and end posts in which the pitch should not exceed 40 times their thickness.

The distance between the edge of any piece and the center of the rivet hole should not be less than $1\frac{1}{4}$ inches for $\frac{3}{4}$-inch and $\frac{7}{8}$-inch rivets except in bars less than $2\frac{1}{2}$ inches wide; when practicable it

* Given by Cambria Steel Co.

should, for all sizes, be at least 2 diameters of the rivet and should not exceed 8 times the thickness of the plate.

EXERCISES ON CHAPTER VII

Problem 235. The end plates of a boiler shell are curved out to a radius of 5 ft. If the plates are $\frac{5}{8}$ in. thick, find the tensile stress due to a steam pressure of 175 lb./in.^{2}

Problem 236. If the thickness of the end plates in Problem 235 is changed to $\frac{1}{2}$ in., the steam pressure being the same, to what radius should they be curved in order that the tensile stress in them shall remain the same?

Problem 237. In a double-riveted lap joint the plates are $\frac{1}{2}$ in. thick, rivets $\frac{7}{8}$ in. in diameter, and pitch 3 in. Calculate the efficiency of the joint.

Problem 238. A boiler shell is to be 4 ft. in diameter, with double-riveted lap joints, and is to carry a steam pressure of 90 lb./in.^{2} with a factor of safety of 5. Determine the thickness of shell plates, and diameter and pitch of rivets. Also calculate the efficiency of the joint.

Problem 239. A cylindrical standpipe is 75 ft. high and 25 ft. inside diameter, with double-riveted, two-strap butt joints. Determine the required thickness of plates near the bottom for a factor of safety of 5, and also the diameter and pitch of rivets.

Problem 240. The cylinder of an hydraulic press is 12 in. in diameter. Find its thickness to stand a pressure of 1500 lb./in.^{2}, if it is made of cast iron and the factor of safety is 10.

Problem 241. A high-pressure, cast-iron water main is 4 in. inside diameter and carries a pressure of 800 lb./in.^{2} Find its thickness for a factor of safety of 15.

Problem 242. The water chamber of a fire engine has a spherical top 18 in. in diameter, and carries a pressure of 250 lb./in.^{2} It is made of No. 7 B. and S. gauge copper, which is reduced in manufacture to a thickness of about .1 in. Determine the factor of safety.

Problem 243. A cast-iron ring 3 in. thick and 8 in. wide is forced on a steel shaft 10 in. in diameter. Find the stresses in ring and shaft, the pressure required to force the ring on the shaft, and the torsional resistance of the fit.

NOTE. Since the ring in this case is relatively thin, assume an allowance of about half the amount given by Moore's formula. Then having given $D_2 = 10$ in., $D_3 = 13$ in., and computed the allowance K, we have also $D_1 = D_2 - K$, and inserting these values in the formulas of Article 121, the required quantities may be found, as explained in Problem 234.

Problem 244. The following data are taken from Stewart's experiments on the collapse of thin tubes under external pressure, the tubes used for experiment being lap-welded, steel boiler flues. Compute the collapsing pressure from the rational formula for thin tubes, given in Article 120, for both the average thickness and least thickness, and note that these two results lie on opposite sides of the value obtained directly by experiment.

Outside Diameter in Inches			Thickness h in Inches			Actual Collapsing Pressure lb./in.²
Average	At place of collapse		Average	At place of collapse		
	Greatest = D	Least = d		Greatest	Least	
8.604	8.610	8.580	0.219	0.230	0.210	870
8.664	8.670	8.625	0.226	0.227	0.204	840
8.665	8.670	8.660	0.212	0.240	0.211	880
8.653	8.665	8.590	0.208	0.220	0.203	970
8.688	8.715	8.605	0.274	0.280	0.266	1430
8.664	8.695	8.635	0.258	0.261	0.248	1320
8.645	8.665	8.635	0.263	0.268	0.259	1590
8.674	8.675	8.675	0.273	0.282	0.270	2030
8.638	8.645	8.615	0.279	0.298	0.280	2200
10.055	10.180	9.950	0.157	0.182	0.150	210

Problem 245. The following data are taken from Stewart's experiments on the collapse of thick tubes under external pressure. The ultimate compressive strength of the material was not given by the experimenter, but from the other elastic properties given, it is here assumed to be $u_c = 38{,}500$ lb./in.² Compute the collapsing pressure from the rational formula for thick tubes, given in Article 120, for both average and least thickness, and compare these results with the actual collapsing pressure obtained by experiment.

Outside Diameter in Inches			Thickness h in Inches			Actual Collapsing Pressure lb./in.²
Average	At place of collapse		Average	At place of collapse		
	Greatest = D	Least = d		Greatest	Least	
4.010	4.020	3.980	0.173	0.203	0.140	2050
4.014	4.050	3.990	0.178	0.277	0.158	2225
4.012	4.050	3.960	0.173	0.200	0.170	2425
4.018	4.050	4.010	0.184	0.192	0.165	2540
2.997	3.010	2.980	0.147	0.151	0.138	3350
2.987	3.010	2.970	0.139	0.139	0.125	2575
2.990	3.010	2.970	0.190	0.218	0.166	4200
2.996	3.020	2.980	0.191	0.216	0.176	4200
2.997	3.020	2.960	0.190	0.215	0.161	4175
3.000	3.020	2.960	0.182	0.192	0.165	3700

Problem 246. A boiler shell $\frac{3}{8}$ in. thick and 5 ft. in diameter has longitudinal, single-riveted lap joints, with 1-in. rivets and $2\frac{1}{4}$-in. rivet pitch. Calculate the maximum steam pressure which can be used with a factor of safety of 5.

Problem 247. A cylindrical standpipe 80 ft. high and 20 ft. inside diameter is made of $\frac{1}{2}$-in. plates at the base with longitudinal, double-riveted, two-strap butt joints, connected by 1-in. rivets with a pitch of $3\frac{1}{2}$ in. Compute the factor of safety when the pipe is full of water.

Problem 248. In a single-riveted lap joint calculate the pitch of the rivets and the distance from the center of the rivets to the edge of the plate under the assumption that the diameter of the rivet is twice as great as the thickness of the plate.

Solution. Consider a strip of width equal to the rivet pitch, i.e. a strip containing one rivet. Let q denote the shearing strength of the rivet, and p the tensile of the plate. Then if h denotes the thickness of the plate, in order that the shearing strength of the rivet may be equal to the tensile strength of the plate along the line of rivet holes, we must have

$$\frac{\pi d^2}{4} q = (c - d)\,hp.$$

Since the rivet is usually of better material than the plate, we may assume that the ultimate shearing strength of the rivet is equal to the ultimate tensile strength of the plate, i.e. assume that $p = q$. Under this assumption the above relation becomes

$$\frac{\pi d^2}{4} = (c - d)\,h = (c - d)\,\frac{d}{2},$$

whence

$$c = 2.5\,d, \text{ approximately.}$$

Similarly, in order that the joint may be equally secure against shearing off the rivet and pulling it out of the plate, i.e. shearing the plate in front of the rivet, the condition is

$$\frac{\pi d^2}{4} q = 2\left(a - \frac{d}{2}\right) hq',$$

where a denotes the "margin," or distance from center of rivets to edge of plate, and q' denotes the ultimate shearing strength of the plate. Assuming that $q' = \frac{4}{5} q$ and $h = \frac{d}{2}$, and solving the resulting expression for a, we have

$$a = 1.5\,d.$$

CHAPTER VIII

FLAT PLATES

123. Theory of flat plates. The analysis of stress in flat plates is, at present, the most unsatisfactory part of the strength of materials. Although flat plates are of frequent occurrence in engineering constructions, as, for example, in manhole covers, cylinder ends, floor panels, etc., no general theory of such plates has as yet been given. Each form of plate is treated by a special method, which, in most cases, is based upon an arbitrary assumption as to the dangerous section, or the reactions of the supports, and therefore leads to questionable results.

Although the present theory of flat plates is plainly inadequate, it is, nevertheless, of value in pointing out the conditions to which such plates are subject, and furnishing a rational basis for the estimation of their strength. The formulas derived in the following paragraphs, if used in this way, with a clear understanding of their approximate nature, will be found to be invaluable in designing, or determining the strength of flat plates.

The following has come to be the standard method of treatment, and is chiefly due to Bach.*

124. Maximum stress in homogeneous circular plate under uniform load. Consider a flat, circular plate of homogeneous material, which bears a uniform load of amount w per unit of area, and suppose that the edge of the plate rests freely on a circular rim slightly smaller than the plate, every point of the rim being maintained at the same level. The strain in this case is greater than if the plate was fixed at the edges, and, consequently, the formula deduced will give the maximum stress in all cases.

* For an approximate method of solution see article by S. E. Slocum entitled "The Strength of Flat Plates, with an Application to Concrete-Steel Floor Panels," *Engineering News*, July 7, 1904.

Now suppose a diametral section of the plate taken, and regard either half of the plate as a cantilever (Fig. 118). Then if r is the radius of the plate, the total load on this semicircle is $\frac{\pi r^2}{2} w$, and its resultant is applied at the center of gravity of the semicircle, which is at a distance of $\frac{4\,r}{3\,\pi}$ from AB. The moment of this resultant about the support AB is therefore $\frac{\pi r^2}{2} w \cdot \frac{4\,r}{3\,\pi}$, or $\frac{2\,r^3 w}{3}$. Similarly, the resultant of the supporting forces at the edge of the plate is of amount $\frac{\pi r^2}{2} \cdot w$, and is applied at the center of gravity of the semi-circumference, which is at a distance of $\frac{2\,r}{\pi}$ from AB. The moment of this resultant about AB is therefore $\frac{\pi r^2 w}{2} \cdot \frac{2\,r}{\pi}$, or $r^3 w$. Hence the total external moment M at the support is

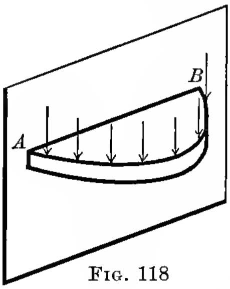


Fig. 118

$$M = r^3 w - \frac{2\,r^3 w}{3} = \frac{r^3 w}{3}.$$

Now assume that the stress at any point of the plate is independent of the distance of this point from the center. Under this arbitrary assumption the stress in the plate is given by the fundamental formula in the theory of beams, namely,

$$p = \frac{Me}{I}.$$

If the thickness of the plate is denoted by h, then, since the breadth of the section is $b = 2\,r$,

$$I = \frac{bh^3}{12} = \frac{rh^3}{6}, \quad \text{and} \quad e = \frac{h}{2}.$$

Consequently,

$$p = \frac{Me}{I} = \frac{\frac{r^3 w}{3} \cdot \frac{h}{2}}{\frac{rh^3}{6}};$$

whence

(83) $$p = w\left(\frac{r}{h}\right)^2.$$

$p = 300 \times \left(\frac{4}{\frac{1}{8}}\right)^2 = 9\ldots \times 300 =$ 1,000 in

C I, in cylinder head, ?!

Föppl has shown that the arbitrary assumption made in deriving this formula can be avoided, and the same result obtained, by a more rigorous analysis than the preceding; and Bach has verified the formula experimentally. Formula (**83**) is therefore well established both theoretically and practically.

Problem 249. The cylinder of a locomotive is 20 in. internal diameter. What must be the thickness of the steel end plate if it is required to withstand a pressure of 160 lb./in.^{2} with a factor of safety of 6?

Problem 250. A circular cast-iron valve gate $\frac{1}{2}$ in. thick closes an opening 6 in. in diameter. If the pressure against the gate is due to a water head of 150 ft., what is the maximum stress in the gate?

125. Maximum stress in homogeneous circular plate under concentrated load. Consider a flat, circular plate of homogeneous material, and suppose that it bears a single concentrated load P which is distributed over a small circle of radius r_0 concentric with the plate. Taking a section through the center of the plate and regarding either half as a cantilever, as in the preceding article, the total rim pressure is $\frac{P}{2}$, and it is applied at a distance of $\frac{2r}{\pi}$ from the center. The total load on the semicircle of radius r_0 is $\frac{P}{2}$, and it is applied at a distance of $\frac{4r_0}{3\pi}$ from the section. Therefore the total external moment M at the section is

$$M = \frac{Pr}{\pi} - \frac{2Pr_0}{3\pi} = \frac{Pr}{\pi}\left(1 - \frac{2r_0}{3r}\right).$$

Assuming that the stress is uniformly distributed throughout the plate, the stress due to the external moment M is given by the formula

$$p = \frac{Me}{I}.$$

If the thickness of the plate is denoted by h, then

$$I = \frac{rh^3}{6} \quad \text{and} \quad e = \frac{h}{2}.$$

Therefore

$$p = \frac{Me}{I} = \frac{\frac{Pr}{\pi}\left(1 - \frac{2r_0}{3r}\right)\frac{h}{2}}{\frac{rh^3}{6}};$$

whence

$$\textbf{(84)} \qquad \boldsymbol{p = \frac{3P}{\pi h^2}\left(1 - \frac{2r_0}{3r}\right)}.$$

If $r_0 = 0$, that is to say, if the load is assumed to be concentrated at a single point at the center of the plate, formula (**84**) becomes

$$p = \frac{3P}{\pi h^2}. \tag{85}$$

If the load is uniformly distributed over the entire plate, then $r_0 = r$ and $P = \pi r^2 w$, where w is the load per unit of area. In this case formula (**84**) becomes

$$p = \frac{3\,\pi r^2 w}{\pi h^2}\left(1 - \frac{2}{3}\right) = w\left(\frac{r}{h}\right)^2,$$

which agrees with the result of the preceding article.

Problem 251. Show that the maximum concentrated load which can be borne by a circular plate is independent of the radius of the plate.

126. Dangerous section of elliptical plate. Consider a homogeneous elliptical plate of semi-axes a and b and thickness h, and suppose that an axial cross is cut out of the plate, composed of two strips AB and CD, each of unit width, and intersecting in the center of the plate, as shown in Fig. 119.

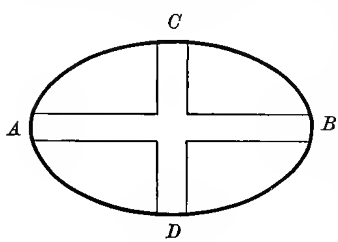


FIG. 119

Now suppose that a single concentrated load acts at the intersection of the cross and is distributed to the supports in such a way that the two beams AB and CD each deflect the same amount at the center. Since AB is of length $2a$, from Article 67, Problem 119, the deflection at the center of AB is $D_1 = \frac{P(2a)^3}{48\,EI}$. From symmetry, the reactions at A and B are equal. Therefore, if each of these reactions is denoted by R_1, $2R_1 = P$, and, consequently,

$$D_1 = \frac{R_1 a^3}{3\,EI}.$$

Similarly, if R_2 denotes the equal reactions at C and D, the deflection D_2 of CD at its center is

$$D_2 = \frac{R_2 b^3}{3\,EI}.$$

If the plate remains intact, the two strips AB and CD must deflect the same amount at the center. Therefore $D_1 = D_2$, and hence

$$\frac{R_1}{R_2} = \frac{b^3}{a^3}. \tag{86}$$

For the beam AB of length $2\,a$ the maximum external moment is $R_1 a$. Also, since AB is assumed to be of unit width, $I = \frac{h^3}{12}$ and $e = \frac{h}{2}$. Hence the maximum stress p' in AB is

$$p' = \frac{Me}{I} = 6\,R_1 \frac{a}{h^2}.$$

Similarly, the maximum stress p'' in CD is

$$p'' = 6\,R_2 \frac{b}{h^2}.$$

Consequently,

$$\frac{p'}{p''} = \frac{R_1 a}{R_2 b},$$

or, since from equation (86) $\frac{R_1}{R_2} = \frac{b^3}{a^3}$,

$$\frac{p'}{p''} = \frac{b^2}{a^2}.$$

By hypothesis, $a > b$. Therefore $p'' > p'$; that is to say, the maximum stress occurs in the strip CD, or in the direction of the shorter axis of the ellipse. In an elliptical plate, therefore, rupture may be expected to occur along a line parallel to the major axis, a result which has been confirmed by experiment.

127. Maximum stress in homogeneous elliptical plate under uniform load. The method of finding the maximum stress in an elliptical plate is to consider the two limiting forms of an ellipse, namely, a circle and a strip of infinite length, and express a continuous relation between the stresses for these two limiting forms. The method is therefore similar to that used in Article 88 in obtaining the modified form of Euler's column formula.

Consider first an indefinitely long strip with parallel sides, supported at the edges and bearing a uniform load of amount w per unit of area. Let the width of the strip be denoted by $2\,b$, and its thickness

by h. Then, if this strip is cut into cross strips of unit width, each of these cross strips can be regarded as an independent beam, the load on one of these unit cross strips being $2\,bw$, and the maximum moment at the center being $\frac{(2\,b)^2\,w}{8}$. Consequently, the maximum stress in the cross strips, and therefore in the original strip, is

$$\text{(87)} \qquad p = \frac{Me}{I} = \frac{\frac{4\,b^2w}{8}\cdot\frac{h}{2}}{\frac{h^3}{12}} = 3\,\frac{b^2w}{h^2}.$$

In the preceding article it was shown that the maximum stress in an elliptical plate occurs in the direction of the minor axis. Therefore equation (**87**) gives the limiting value which the stress in an elliptical plate approaches as the ellipse becomes more and more elongated.

For a circular plate of radius b and thickness h the maximum stress was found to be

$$\text{(88)} \qquad p = \frac{b^2w}{h^2}.$$

Comparing equations (**87**) and (**88**), it is evident that the maximum stress in an elliptical plate is given, in general, by the formula

$$p = k\,\frac{b^2w}{h^2},$$

where k is a constant which lies between 1 and 3. Thus, for $\frac{b}{a} = 1$, that is, for a circle, $k = 1$; whereas, if $\frac{b}{a} = 0$, that is, for an infinitely long ellipse, $k = 3$. The constant k may therefore be assumed to have the value

$$k = 3 - 2\,\frac{b}{a},$$

which reduces to the values 1 and 3 for the limiting cases, and in other cases has an intermediate value depending on the form of the plate. Consequently,

$$\text{(89)} \qquad \boldsymbol{p = \left(3 - 2\,\frac{b}{a}\right)\frac{b^2w}{h^2} = \frac{(3\,a - 2\,b)\,b^2w}{ah^2},}$$

which is the required formula for the maximum stress p in a homogeneous elliptical plate of thickness h and semi-axes a and b.

Problem 252. A cast-iron manhole cover 1 in. thick is elliptical in form, and covers an elliptical opening 3 ft. long and 18 in. wide. How great a uniform pressure will it stand?

128. Maximum stress in homogeneous square plate under uniform load. In investigating the strength of square plates the method of taking a section through the center of the plate and regarding the portion of the plate on one side of this section as a cantilever is used, but experiment is relied upon to determine the position of the dangerous section. From numerous experiments on flat plates, Bach has found that homogeneous square plates under uniform load always break along a diagonal.*

Consider a homogeneous square plate of thickness h and side $2a$, which bears a uniform load w per unit of area. Suppose that a diagonal section of this plate is taken, and consider either half as a cantilever, as shown in Fig. 120. Then the total load on the plate is $4wa^2$, and the reaction of the support under each edge is wa^2. If d denotes the length of the diagonal AC, the resultant pressure on each edge of the plate is applied at a distance $\frac{d}{4}$ from AC, and therefore the moment of these resultants

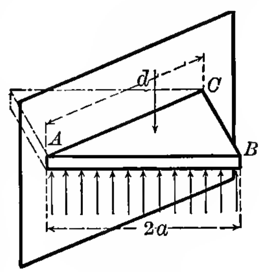


Fig. 120

about AC is $2(wa^2)\frac{d}{4}$, or $\frac{wa^2d}{2}$. The total load on the triangle ABC is $2wa^2$, and its resultant is applied at the center of gravity of the triangle, which is at a distance of $\frac{d}{6}$ from AC. Therefore the moment of the load about AC is $(2wa^2)\frac{d}{6}$, or $\frac{wa^2d}{3}$. Therefore the total external moment M at the section AC is

$$M = \frac{wa^2d}{2} - \frac{wa^2d}{3} = \frac{wa^2d}{6}.$$

* Bach, *Elasticität u. Festigkeitslehre*, 3d ed., p. 561.

Hence the maximum stress in the plate is

$$p = \frac{Me}{I} = \frac{\frac{wa^2d}{6} \cdot \frac{h}{2}}{\frac{dh^3}{12}},$$

from which

$$\textbf{(90)} \qquad p = w\left(\frac{a}{h}\right)^2.$$

The maximum stress in a square plate of side $2a$ is therefore the same as in a circular plate of diameter $2a$.

Problem 253. What must be the thickness of a wrought-iron plate covering an opening 4 ft. square to carry a load of 200 lb./ft.^{2} with a factor of safety of 5?

129. Maximum stress in homogeneous rectangular plate under uniform load. In the case of rectangular plates experiment does not indicate so clearly the position of the dangerous section as it does for square plates. It will be assumed in what follows, however, that the maximum stress occurs along a diagonal of the rectangle. This assumption is at least approximately correct if the length of the rectangle does not exceed two or three times its breadth.

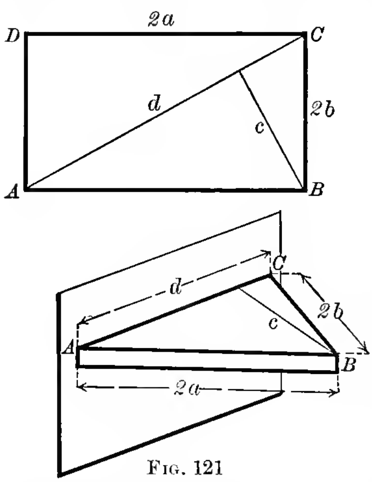


Fig. 121

Let the sides of the rectangle be denoted by $2a$ and $2b$, and the thickness of the plate by h (Fig. 121). Also let d denote the length of the diagonal AC, and c the altitude of the triangle ABC. Now suppose that a diagonal section AC of the plate is taken, and consider the half plate ABC as a cantilever, as shown in Fig. 121. If w denotes the unit load, the total load on the plate is $4abw$, and consequently the resultant of the reactions of the supports along AB and BC is of amount $2abw$, and is applied at a distance $\frac{c}{2}$ from AC. Therefore the moment of the supporting force about AC is $abwc$. Also, the total load on the triangle

ABC is $2\,abw$, and it is applied at the center of gravity of the triangle, which is at a distance of $\frac{c}{3}$ from AC. Consequently, the total moment of the load about AC is $\frac{2\,abwc}{3}$. Therefore the total external moment M at the section AC is

$$M = abwc - \frac{2\,abwc}{3} = \frac{abwc}{3},$$

and the maximum stress in the plate is

$$p = \frac{Me}{I} = \frac{\frac{abwc}{3}\cdot\frac{h}{2}}{\frac{dh^3}{12}} = \frac{2\,wabc}{dh^2},$$

or, since $cd = 4\,ab$,

(91) $$\boldsymbol{p = w\frac{c^2}{2\,h^2},}$$

which gives the required maximum stress.

For a square plate $a = b$ and $c = a\sqrt{2}$, and formula **(91)** reduces to formula **(90)** for square plates, obtained in the preceding article.

Problem 254. A wrought-iron trapdoor is 5 ft. long, 3 ft. wide, and $\frac{3}{8}$ in. thick. How great a uniform load will it bear?

130. Non-homogeneous plates; concrete-steel floor panels. The formulas derived in the preceding articles apply only to flat plates of homogeneous material. If a plate is composed of non-homogeneous material, such as reënforced concrete, the maximum stress is given by the formula

$$p = \frac{Me'}{I'},$$

where I' is the moment of inertia of the equivalent homogeneous section obtained from the non-homogeneous section as explained in Article 48, and e' is the distance of the extreme fiber of this equivalent homogeneous section from its neutral axis.

Thus, from Article 124, the external moment M on half of a uniformly loaded circular plate is $M = \frac{r^3w}{3}$, and, consequently, the maximum stress in a uniformly loaded, non-homogeneous, circular plate is given by the formula

(92) $$p = \frac{r^3we'}{3\,I'},$$

where I' and e' refer to the equivalent homogeneous section as explained above, and this section is taken through the center of the plate.

Similarly, from Article 128, the maximum stress in a uniformly loaded, non-homogeneous, square plate of side $2a$ is given by the formula

(93) $$p = \frac{\sqrt{2}}{3} \cdot \frac{wa^3e'}{I'};$$

and, from Article 129, the maximum stress in a uniformly loaded, non-homogeneous, rectangular plate of sides $2a$ and $2b$ by the formula

(94) $$p = \frac{abwce'}{3I'},$$

in which e' and I' refer to the equivalent homogeneous section obtained from a diagonal section of the plate.

Problem 255. A concrete-steel floor panel is 18 ft. long, 15 ft. wide, and 4 in. thick, and is reënforced by square wrought-iron rods 1 in. thick, placed $\frac{3}{4}$ of an inch from the bottom of the slab and spaced 1 ft. apart. Find the maximum stress in the panel under a total live and dead load of 150 lb./ft.^{2}.

NOTE. Take a diagonal section of the panel and calculate the equivalent homogeneous section corresponding to it. Then find the position of the neutral axis of this equivalent homogeneous section, and its moment of inertia about this neutral axis, as explained in Article 48. The maximum stress can then be obtained from formula (**94**).

Problem 256. Design a floor panel 14 ft. square, to be made of reënforced concrete and to sustain a total uniform load of 120 lb./ft.^{2} with a factor of safety of 4.

EXERCISES ON CHAPTER VIII

Problem 257. The steel diaphragm separating two expansion chambers of a steam turbine is subjected to a pressure of 150 lb./in.^{2} on one side and 80 lb./in.^{2} on the other. Find the required thickness for a factor of safety of 10.

Problem 258. The cylinder of an hydraulic press is made of cast steel, 10 in. inside diameter, with a flat end of the same thickness as the walls of the cylinder. Find the required thickness for a factor of safety of 20. Also find how much larger the factor of safety would be if the end was made hemispherical instead of flat.

Problem 259. The cylinder of a steam engine is 16 in. inside diameter and carries a steam pressure of 125 lb./in.^{2} If the cylinder head is mild steel, find its thickness for a factor of safety of 10.

Problem 260. A cast-iron valve gate 10 in. in diameter is under a pressure head of 200 ft. Find its thickness for a factor of safety of 15.

Problem 261. A cast-iron elliptical manhole cover is 18 in. × 24 in. in size and is designed to carry a concentrated load of 1000 lb. If the cover is ribbed, how thick must it be for a factor of safety of 20, assuming that the ribs double its strength?

Problem 262. Thurston's rule for the thickness of cylinder heads for steam engines is

$$h = .00035\, wD,$$

where h = thickness of head in inches,
D = inside diameter of cylinder in inches,
w = pressure in lb./in.^{2}

Compare this formula with Bach's, assuming the material to be wrought iron, and using the data of Problem 259.

Problem 263. Show that Thurston's rule for thickness of cylinder head, given in Problem 262, makes thickness of head $= 1\frac{1}{4}$ times thickness of walls.

Problem 264. Nichols's rule for the proper thickness of unbraced flat wrought-iron boiler heads is

$$h = \frac{Fw}{10p},$$

where h = thickness of head in inches,
F = area of head in square inches,
w = pressure per square inch,
p = working stress $= \dfrac{44{,}800}{8} = \dfrac{\text{ult. strength in tension}}{\text{factor of safety}}$.

Compare this empirical rule with Bach's formula, using the data of Problem 259 and assuming the material to be wrought iron.

Problem 265. Nichols's rule for the collapsing pressure of unbraced flat wrought-iron boiler heads is

$$w = \frac{10\, hu_t}{F},$$

where w = collapsing pressure in lb./in.^{2},
h = thickness of head in inches,
u_t = ultimate tensile strength in lb./in.^{2},
F = area of head in square inches.

Show that Nichols's two formulas are identical and that therefore they cannot be rational.

Problem 266. The following data are taken from Nichols's experiments on flat wrought-iron circular plates.

DIAMETER IN.	THICKNESS IN.	ACTUAL BURSTING PRESSURE LB./IN.^{2}
34.5	$\frac{9}{16}$	280
34.5	$\frac{3}{8}$	200
28.5	$\frac{3}{8}$	300
26.5	$\frac{3}{8}$	370

Using this data, compare Bach's and Grashof's rational formulas with Nichols's and Thurston's empirical formulas, as given below:

Circular plate, supported at edge and uniformly loaded.

Bach, $h = r\sqrt{\frac{w}{p}} = .5\,D\sqrt{\frac{w}{p}}$,

Grashof, $h = \sqrt{\frac{5\,r^2 w}{6\,p}} = .4564\,D\sqrt{\frac{w}{p}}$,

Nichols, $h = \frac{wF}{10\,p} = .0785\,\frac{wD^2}{p}$,

Thurston, $h = .00035\,wD$,

where $h =$ thickness of head in inches,
$D =$ diameter of head in inches $= 2\,r$,
$w =$ pressure in lb./in.^{2},
$p =$ working stress in lb./in.^{2},
$F =$ area of head in square inches $= \frac{\pi D^2}{4}$.

Note that the Nichols and Thurston formulas apply only to wrought iron.

CHAPTER IX

CURVED PIECES: HOOKS, LINKS, AND SPRINGS

131. Erroneous analysis of hooks and links. In calculating the strength of a curved piece whose axis is a plane curve, such as a hook or a link of a chain, many engineers are accustomed to assume that the distribution of stress is the same as in a straight beam subjected to an equal bending moment and axial load. For example, in calculating the strength of a hook, such as shown in Fig. 122, the practice has been to take a section AB where the bending moment is a maximum, and calculate the unit stress p on AB by the formula

$$p = \frac{P}{F} + \frac{(Pd)e}{I},$$

where the first term denotes the direct stress on the section AB of area F, and the second term represents the bending stress due to a moment Pd calculated from the formula for straight beams.

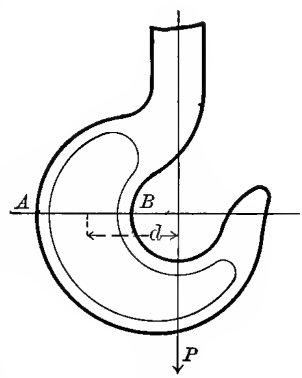


FIG. 122

The bending formula for straight beams, however, does not apply to curved pieces, as will be shown in what follows. Moreover, experiment has conclusively shown that a curved piece breaks at the point of sharpest curvature, whereas the above formula takes no account whatever of the curvature. The above formula is therefore not even approximately correct, and is cited as a popular error against which the student is warned.

132. Bending strain in curved piece. Consider a curved piece which is subjected to pure bending strain, and assume that the axis of the piece is a plane curve and also that the radius of curvature is not very large as compared with the thickness of the piece. Hooke's law and Bernoulli's assumption will be taken as the starting point

for the analysis of stress, as in the theory of straight beams; that is to say, it will be assumed that the stress is proportional to the deformation produced, and that any plane section remains identical with itself during the deformation.

Since the fibers on the convex side are longer than those on the concave side, it will take less stress to deform them an equal amount. Therefore the neutral axis does not pass through the center of gravity G of the section, but through some other point D, below G, as shown in Fig. 123. For if the neutral axis passed through G, the total deformation above and below G would be of equal amount, and therefore the total stress above G would be less than that below G, since the fibers above G are longer than those below. This shifting of the neutral axis constitutes the fundamental difference between the theory of straight and curved pieces.

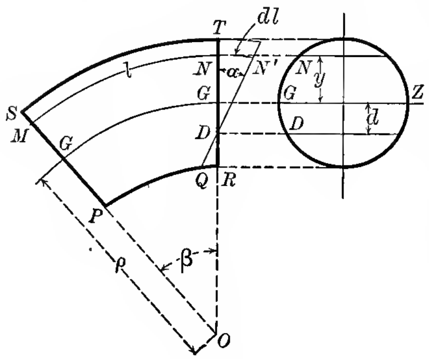


Fig. 123

Now let the length of any fiber, such as MN in Fig. 123, be denoted by l, and the distance of this fiber from a gravity axis GZ by y. Also, let ρ denote the radius of curvature OG of the piece, β the angle between two plane sections, and α the angle of deformation of a plane section. Then

$$l = \beta \cdot MO = (OG + GN)\beta = (\rho + y)\beta,$$

and the deformation dl of the fiber MN is

$$dl = NN' = \alpha \cdot ND = (y + d)\alpha,$$

where d denotes the distance GD between the neutral axis and the gravity axis. From Hooke's law,

$$\frac{dl}{l} = \frac{p}{E};$$

whence

$$p = \frac{Edl}{l} = \frac{E(y + d)\alpha}{(\rho + y)\beta}.$$

Let $\frac{\alpha}{\beta} = k$, where k is a constant. Then this expression for p reduces to

(**95**) $$p = Ek\frac{y+d}{y+\rho}.$$

Under the assumption of pure bending strain the shear is zero and the normal stresses form a couple. Therefore the algebraic sum of the normal stresses is zero; that is to say,

$$\int pdF = 0,$$

or, substituting the value of p from equation (**95**),

$$kE\int\frac{y+d}{y+\rho}\,dF = 0.$$

Since k and E are constants and not zero, the integral must be zero. Therefore, separating the integral into parts,

$$\int\frac{ydF}{y+\rho} + d\int\frac{dF}{y+\rho} = 0;$$

whence

(**96**) $$d = -\frac{\displaystyle\int\frac{ydF}{y+\rho}}{\displaystyle\int\frac{dF}{y+\rho}},$$

which gives the distance of the neutral axis below the center of gravity of the section.

Now let M denote the external bending moment acting on any given section of area F, dF an infinitesimal area taken anywhere in this section, p the stress acting on it, and y its distance from the gravity axis GZ. Then

$$\int(y+d)\,pdF = M,$$

or, substituting the value of p from equation (**95**),

$$Ek\int\frac{(y+d)^2}{y+\rho}\,dF = M;$$

consequently

$$k = \frac{M}{E\displaystyle\int\frac{(y+d)^2}{y+\rho}\,dF},$$

and hence

(**97**) $$p = Ek\frac{y+d}{y+\rho} = \frac{M(y+d)}{(y+\rho)\displaystyle\int\frac{(y+d)^2}{y+\rho}\,dF},$$

which is the required formula for calculating the bending stress at any point of a curved piece.

133. Simplification of formula for unit stress. In formulas (**96**) and (**97**), derived in the preceding article, the integrals involved make the formulas difficult of application. The following geometrical transformation, which is due to Résal,* greatly simplifies the formulas and their application.

The first step is a geometrical transformation of the boundary of the given cross section. Consider a symmetrical cross section, for example the circle shown in Fig. 124, and let OY be an axis of symmetry passing through the center of curvature C of the section, and OZ a gravity axis perpendicular to OY. Now suppose radii drawn from C to each point M in the boundary of the cross section. From H, the point of intersection of CM with the gravity axis OZ, erect a perpendicular to OZ, and from M draw a perpendicular to OY. Then these two perpendiculars will intersect in a point of the transformed boundary, as shown in Fig. 124.

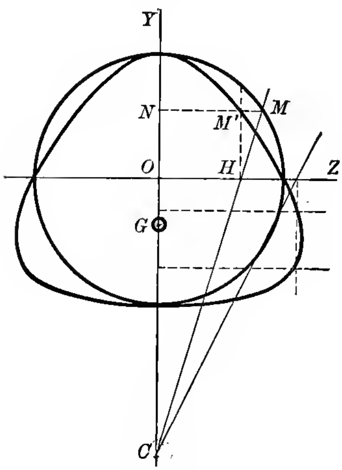


Fig. 124

It will now be proved (1) that the distance of the center of gravity G of the transformed section from the center of gravity O of the original section is the value of d given by formula (**96**), and (2) that the moment of inertia of the transformed section is the integral which occurs in formula (**97**).

In Fig. 124 the distance NM' is the Z-coördinate of the point M'; let it be denoted by z'. Then

$$NM' = z' = OH = MN\frac{CO}{CN} = z\frac{\rho}{\rho + y}.$$

The distance d' of the center of gravity G of the transformed

* *Résistance des Matériaux*, pp. 385 *et seq.*

section below the center of gravity O of the original section is

$$d' = OG = -\frac{\int z'ydy}{\int z'dy} = -\frac{\int zy\frac{\rho}{\rho+y}dy}{\int z\frac{\rho}{\rho+y}dy}.$$

Dividing out the constant ρ and replacing the element of area zdy by dF, this expression for d' becomes

$$d' = -\frac{\int\frac{ydF}{y+\rho}}{\int\frac{dF}{y+\rho}},$$

which is identical with the value of d given by formula (**96**) above. Consequently, *the neutral axis of the original cross section coincides with the gravity axis of the transformed section.*

Now let the moment of inertia of the transformed section be denoted by I'. Then

$$I' = \int y'^2 dF',$$

in which y' is measured from the gravity axis of the *transformed* section, that is, from a line through G parallel to OZ; and dF' denotes an element of area of the transformed section; whence $dF' = z'dy'$. Therefore, since

$$y' = y + d,\quad z' = z\frac{\rho}{\rho + y},\quad \text{and}\quad dy' = dy,$$

the expression for I' becomes

$$I' = \int (y+d)^2\left(z\frac{\rho}{\rho+y}\right)dy;$$

or, if the element of area zdy is denoted by dF,

$$I' = \rho\int\frac{(y+d)^2 dF}{\rho + y}.$$

This integral, however, is the one which occurs in formula (**97**). Consequently, if its value from the above equation is substituted in (**97**), the expression for the unit stress p simplifies into

$$\textbf{(98)}\qquad p = \frac{M\rho}{I'}\cdot\frac{y+d}{y+\rho}.$$

For an ordinary beam without initial curvature, $d = 0$, $I' = I$, and $\rho \doteq \infty$, in which case, since $\dfrac{\rho}{y+\rho} = 1 - \dfrac{y}{\rho + y}$, the formula reduces to the ordinary beam formula $p = \dfrac{My}{I}$.

To avoid the confusion which may arise from positive and negative values of y in applying formula (**98**), note that

$$\frac{y+d}{y+\rho} = \frac{\text{distance of fiber from neutral axis}}{\text{distance of same fiber from center of curvature}}.$$

This quotient is then an abstract number, and its substitution in formula (**98**) gives the numerical value of the stress p without regard to sign.

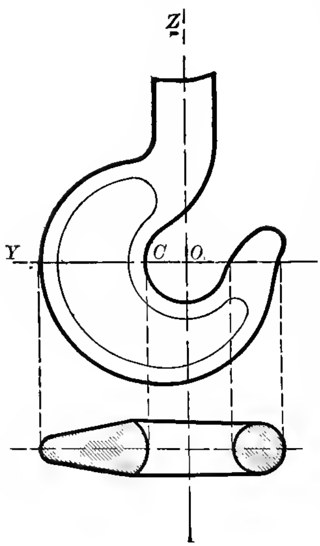


FIG. 125

The following problem illustrates the application of the formula.

Problem 267. The wrought-iron crane hook, shown in Fig. 125, is designed to support a load of ten tons. Find the maximum stress in the hook under this load, and thence determine the factor of safety.

Solution. Let a cross section OCY of the hook be taken at the position of maximum moment, as shown in the shaded projection in Fig. 125.

In Fig. 126 let the curve numbered 1 represent this projection. The gravity axis DF of this section, perpendicular to the axis of symmetry COY, is first determined, which may be done by the graphical method explained in Article 47, or otherwise.* Curve 1 is then transformed into curve 2 by the method explained in Article 133, the light construction lines on the left of OY showing how this is accomplished.

The moment of inertia I' of curve 2 is then found graphically by the method explained in Article 47. This method consists in first transforming curve 2 into curves 3 and 4, as there explained, then measuring the areas between OY and curves 3 and 4 by means of a planimeter, and finally substituting the areas so found in the formulas for the moment of inertia I' of curve 2 and the distance c of its center of gravity from AB, given in Article 47.

In the present case we have then the following numerical values for substitution :

$\rho = CE = 4.4$ in., $F = 7.9$ in.^{2}, $CO = 2.2$ in., $EO' = 2.8$ in.,
$I' = 14.1$ in.^{4}, $c = 1.8$ in., $OE = 2.2$ in., $d = 0.4$ in.

$$M = 20{,}000 \times 4.4 = 88{,}000 \text{ in. lb.}$$

* A simple method of determining a gravity axis sufficiently accurate for ordinary purposes consists in cutting the section out of cardboard and balancing it on a knife edge.

Consequently, at the outer fiber O' we have

$$p = \frac{88{,}000 \times 4.4}{14.1} \times \frac{2.8 + 0.4}{2.8 + 4.4} = 12{,}200 \text{ lb./in.}^2 \text{ compression; and at the inner fiber } O,$$

$$p = \frac{88{,}000 \times 4.4}{14.1} \times \frac{-2.2 + 0.4}{-2.2 + 4.4} = -22{,}470 \text{ lb./in.}^2, \text{ tension.}$$

Moreover, the direct tensile stress on the cross section is $\dfrac{20{,}000}{7.9} = 2530$ lb./in.^{2} Hence the actual total stress on the outer fiber O' is $12{,}200 - 2530 = 9670$ lb./in.^{2} compression, corresponding to a factor of safety of about 5; and on the inner fiber O is $22{,}470 + 2530 = 25{,}000$ lb./in.^{2} tension, corresponding to a factor of safety of 2.

Problem 268. By the formula given in Article 131, calculate the maximum bending stress and the maximum total stress on the hook shown in Fig. 125, and compare the results with those of the preceding problem.

Problem 269. The dangerous section of a hook similar to that shown in Fig. 125 has for its dimensions $b = 2\frac{3}{4}$ in., $h = 6$ in., $r_1 = 1\frac{3}{4}$ in., $r_2 = \frac{5}{8}$ in. (Fig. 127), and $OC = 2\frac{5}{8}$ in. (Fig. 126). Using a factor of safety of 4, find the safe load for the hook.

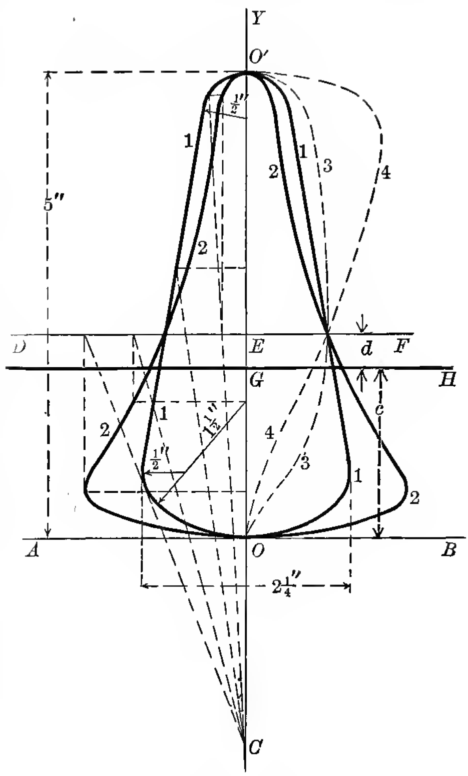


Fig. 126

For all practical purposes the theory of stress in curved pieces here presented is undoubtedly the most satisfactory theory which has yet been developed. A more rigorous analysis of the subject, however, introducing Poisson's ratio of lateral deformation, has been given by Andrews and Pearson in their monograph on crane and coupling hooks.* Although this discussion is extremely valuable from a theoretical standpoint, it has been shown that its results exhibit but a slight refinement over the simpler discussion given above, — a difference considerably less than the variation which may be expected in the physical properties of materials used commercially.† By reason of

* Karl Pearson and E. H. Andrews, "Theory in Crane and Coupling Hooks," etc., *Tech. Series I; Drapers Co. Research Memoirs* [Dulau and Co., 37 Soho Square, London, W.]. See also *Am. Mach.*, Vol. XXXII, Oct. 7, 1909, pp. 615–619; Dec. 16, 1909, pp. 1065–1067.

† *Am. Mach.*, Vol. XXXIII, Nov. 24, 1910, pp. 954–955.

this uncertainty as to the exact values of the physical constants involved, the simpler method is of more value to the designer. The enormous amount of labor and liability to error involved in the application of the Pearson-Andrews formula is, in fact, prohibitive where speed and accuracy are an object.

* **134. Curved piece of rectangular cross section.** If the cross section of a curved piece is rectangular, the integrals in formulas (**96**) and (**97**), Article 132, can be easily evaluated. These formulas may therefore be used for calculating the strength of the piece in preference to the graphical method explained in the preceding article.

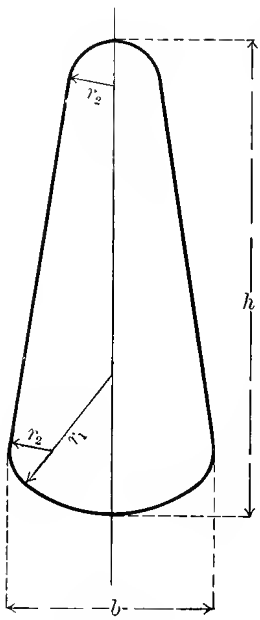


FIG. 127

Let the cross section of the piece be a rectangle of breadth b and depth h, and let ρ denote the radius of curvature of the piece at the section under consideration. From formula (**96**), the distance of the neutral axis of the section from the mean fiber, or gravity axis, is

$$d = -\frac{\int \frac{ydF}{y+\rho}}{\int \frac{dF}{y+\rho}},$$

where y denotes the distance of the infinitesimal area dF from the gravity axis. In the present case $dF = bdy$; hence

$$d = -\frac{b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{ydy}{y+\rho}}{b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dy}{y+\rho}} = -\frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{ydy}{y+\rho}}{\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dy}{y+\rho}}.$$

By division, $\frac{y}{y+\rho} = 1 - \frac{\rho}{y+\rho}$. Consequently, the numerator of the above fraction becomes

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{ydy}{y+\rho} = \int_{-\frac{h}{2}}^{\frac{h}{2}} dy - \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dy}{y+\rho} = h - \rho \log_e \frac{2\rho + h}{2\rho - h}.$$

* For a brief course the remainder of this chapter may be omitted.

Similarly, the denominator becomes

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dy}{y+\rho} = \log_e (y+\rho)\Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \log_e \frac{2\,\rho + h}{2\,\rho - h}.$$

Consequently,

$$d = -\frac{h - \rho \log_e \dfrac{2\,\rho + h}{2\,\rho - h}}{\log_e \dfrac{2\,\rho + h}{2\,\rho - h}},$$

which may be written

(**99**) $$d = \rho - \frac{h}{\log_e \dfrac{2\,\rho + h}{2\,\rho - h}}.$$

From formula (**97**), Article 132, the unit stress p at any point in the cross section, distant y from the mean fiber, is given by the equation

$$p = \frac{M(y+d)}{(y+\rho)\displaystyle\int \frac{(y+d)^2}{y+\rho}\,dF}.$$

Replacing dF by bdy, and separating the integral in the denominator into partial integrals by means of division, this integral becomes

$$\begin{aligned}
\int \frac{(y+d)^2}{y+\rho}\,dF &= \int \frac{(y^2 + 2\,dy + d^2)\,bdy}{y+\rho} \\
&= b\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} ydy + \int_{-\frac{h}{2}}^{\frac{h}{2}} (2\,d - \rho)\,dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{(d-\rho)^2 dy}{y+\rho}\right] \\
&= b\left[\frac{y^2}{2} + (2\,d - \rho)\,y + (d-\rho)^2 \log_e (y+\rho)\right]\Bigg|_{-\frac{h}{2}}^{\frac{h}{2}} \\
&= b\left[(2\,d - \rho)h + (d-\rho)^2 \log_e \frac{2\,\rho + h}{2\,\rho - h}\right].
\end{aligned}$$

Substituting for d its value from equation (**99**), this expression finally reduces to

$$\int \frac{(y+d)^2}{y+\rho}\,dF = bh\left[\rho - \frac{h}{\log_e \dfrac{2\,\rho + h}{2\,\rho - h}}\right] = bhd.$$

Hence the expression for p becomes

$$p = \frac{M(y + d)}{(y + \rho) bhd}. \tag{100}$$

The stresses on the extreme fibers are the values of p for $y = \pm \frac{h}{2}$. Hence

$$p_{\max} = \frac{M(h \pm 2d)}{(h \pm 2\rho) bhd}. \tag{101}$$

Note that the stress on the inside fiber is always negative, in consequence of which the sign of M should be negative if it tends to decrease the radius of curvature, and vice versa.

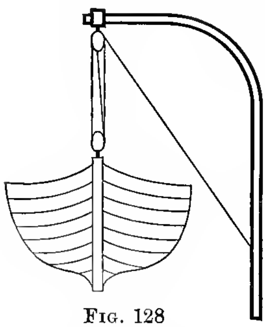


FIG. 128

Problem 270. A boat's davits are composed of two wrought-iron bars $2\frac{1}{2}$ in. square, bent to a radius of 2 ft., as shown in Fig. 128. If the boat weighs 500 lb. and is hung $3\frac{1}{2}$ ft. from the vertical axis of the davits, find the maximum stress in the davits and the factor of safety.

135. Effect of sharp curvature on bending strength. Consider a sharply curved prismatic piece which is subjected to bending strain. From the above discussion, it is known that for a section taken in the neighborhood of the bend, the neutral axis does not coincide with the gravity axis but approaches the center of curvature. The neutral fiber is therefore separated from the mean fiber, or axis of the piece, and takes some such position as that shown by the broken line in Fig. 129. Consequently the inner fiber through A must endure a far greater stress than that deduced from formulas for the straight portion. Engineers and constructors have learned by experience that sharp curvature produces weakness of this kind, and that it is necessary to reënforce a piece at a bend either by increasing its diameter or by adding a brace.

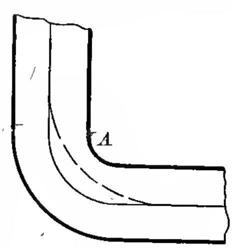


FIG. 129

As an illustration of the effect of sharp curvature on bending strength, suppose that a bar of rectangular cross section is bent into a right angle, as shown in Fig. 130. In this case the center of curvature

of the mean fiber BC is at A. Therefore, if h denotes the thickness of the piece, the radius of curvature of BC is $\rho = \frac{h}{2}\cdot$ Consequently,

$$\log_e \frac{2\rho + h}{2\rho - h} = \log_e \frac{2h}{0} = \infty,$$

and hence formula (**99**) becomes

$$d = \rho = \frac{h}{2}\cdot$$

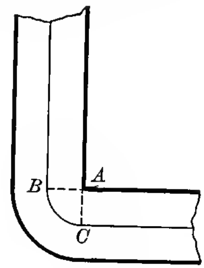


FIG. 130

Therefore the neutral fiber passes through the vertex of the angle A, and consequently a piece of this kind can offer no resistance to bending. In other words, if a piece is bent exactly at right angles on itself, the slightest bending strain must produce incipient rupture.

This example is useful, then, in pointing out the danger of sharp curvature and showing how rapidly the strength decreases with the radius of curvature.

136. Maximum moment in circular piece. Consider a prismatic piece with a circular axis, such as a ring or a section of pipe, and suppose that it is subjected to two equal and opposite forces P, either of tension or compression, acting along a diameter as shown in Fig. 131. Draw a second diameter MN at right angles to the direction in which the forces P act. Since these two diameters divide the figure into four symmetrical parts, it is only necessary to consider one of these parts, say the upper left-hand quadrant. The forces acting on any section of this quadrant consist of a single force and a moment.

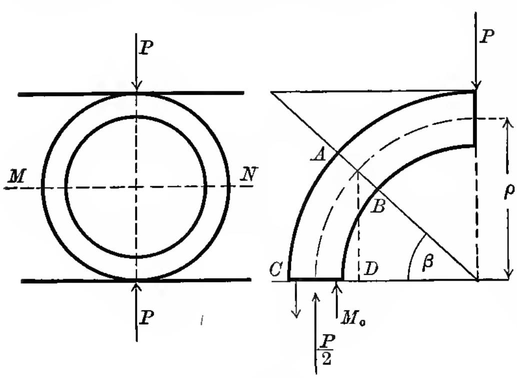


FIG. 131

On the base CD of the quadrant this single force is of amount $\frac{P}{2}$,

and the unknown moment will be denoted by M_0. On any other section AB the bending moment M and single force P' are respectively

$$(102) \qquad \begin{aligned} M &= M_0 + \frac{P}{2}(\rho - \rho\cos\beta), \\ P' &= \frac{P}{2}\cos\beta, \end{aligned}$$

in which ρ is the radius of the mean fiber and β is the angle which the plane of the section AB makes with the base CD.

Now, no matter whether the section is flattened or elongated by the strain, from the symmetry of the figure the diametral sections MN and PP will always remain at right angles to one another. Therefore the total angular deformation $\Delta\beta$ for the quadrant under consideration must be zero; that is to say,

$$\int_0^{\frac{\pi}{2}} d\beta = 0.$$

But, from Article 67,

$$d\beta = \frac{Mdx}{EI}.$$

Consequently,

$$\int_0^{\frac{\pi}{2}} \frac{Mdx}{EI} = 0.$$

Inserting in this expression the value of M obtained above,

$$\int_0^{\frac{\pi}{2}} \left(\frac{M_0 + \frac{P\rho}{2} - \frac{P\rho}{2}\cos\beta}{EI} \right) \rho d\beta = 0,$$

or

$$\int_0^{\frac{\pi}{2}} \left(M_0 + \frac{P\rho}{2} - \frac{P\rho}{2}\cos\beta \right) d\beta = M_0\frac{\pi}{2} + \frac{P\rho\pi}{4} - \frac{P\rho}{2} = 0;$$

whence

$$M_0 = -\frac{\pi - 2}{2\pi} P\rho = -.182\ P\rho,$$

which is the maximum negative moment.

From formula (**102**), the maximum positive moment must occur when $\cos\beta = 0$, that is, when $\beta = \frac{\pi}{2}$, or at top and bottom. Therefore

$$M_{\max} = M_{\frac{\pi}{2}} = \frac{P\rho}{\pi} = .318\ P\rho.$$

The maximum moment, therefore, occurs at the points of application of the forces. From formula (**102**), the direct stress at these points is zero.

Having determined the position and amount of the maximum bending moment, the maximum bending stress can be calculated by the graphical method explained in Article 133, or, if the piece is rectangular in section, by formulas (**99**) and (**100**) or (**101**) in Article 134.

Problem 271. A wrought-iron anchor ring is 6 in. in inside diameter and 2 in. in sectional diameter. With a factor of safety of 4, find by the graphical method of Article 133 the maximum pull which the ring can withstand.

Problem 272. A cast-iron pipe 18 in. in internal diameter and 1 in. thick is subjected to a pressure of 150 lb./linear foot at the highest point of the pipe. Find the maximum stress in the pipe.

HINT. Use formula (**101**), Article 134.

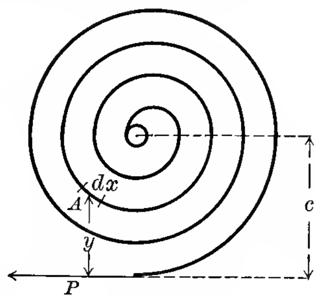


FIG. 132

137. Plane spiral springs. Consider a plane spiral spring, such as the spring of a clock or watch. Let P denote the force tending to wind up the spring, and c the perpendicular distance of P from the spindle on which the spring is wound (Fig. 132). Also, let dx denote a small portion of the spring at any point A distant y from P. Then the moment at A is $M = Py$; and hence, from Article 67, the angular deformation $d\beta$ for the portion dx is given by the formula

$$d\beta = \frac{Mdx}{EI} = \frac{Pydx}{EI}.$$

Therefore the total angular deformation of the spring is

$$\beta = \int_0^l d\beta = \int_0^l \frac{Pydx}{EI} = \frac{P}{EI}\int_0^l ydx.$$

Since the average value of y is c, and the integral of dx is the length of the spring l,

$$\int_0^l ydx = cl,$$

and hence

$$\beta = \frac{Pcl}{EI}.$$

The resilience W of the spring is, therefore,

$$W = \frac{1}{2} M\beta = \frac{P^2c^2l}{2\,EI}.$$

If the spring is of rectangular cross section, which is the usual form for plane spiral springs, the stress can be calculated by formulas (**99**) and (**101**), Article 134.

The formula just obtained for the resilience of a spring is a special case of a more general formula. Thus consider a portion of a beam of length $AB = l$, and let M denote the average bending moment over the part considered, and β the change in slope in passing from A to B. Then the work done in bending the portion AB is $W = \frac{1}{2} M\beta$, or, since $\beta = \dfrac{Ml}{EI}$, this becomes $W = \dfrac{M^2l}{2EI}$. In the case of the spring considered above, the mean value of the bending moment was $M = Pc$.

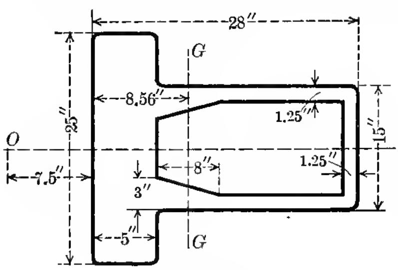


Fig. 133

Furthermore, if p denotes the greatest stress at the elastic limit and e the distance at which it acts from the neutral axis, then $M = \dfrac{pI}{e}$, and consequently the resilience of the beam is

$$W = \frac{p^2lI}{2\,Ee^2}.$$

For the resilience of a piece under direct stress, see Article 22.

Problem 273. A steel clock spring $\frac{3}{4}$ in. wide and $\frac{1}{64}$ in. thick is wound on a spindle $\frac{3}{16}$ in. in diameter. With a factor of safety of 5, what is the maximum moment available for running the mechanism?

Suggestion. The dangerous section occurs at the spindle where the moment is greatest and the radius least. Therefore, in the present case, $\rho = \frac{13}{128}$ in., $h = \frac{1}{64}$ in., $b = \frac{3}{4}$ in. Also, since the ultimate tensile strength of spring steel is about 240,000 lb./in.^{2}, $p_{max} = \dfrac{240{,}000}{5} = 48{,}000$ lb./in.^{2} d can then be calculated by formula (**99**), and M by formula (**101**).

EXERCISES ON CHAPTER IX

Problem 274. A flat spiral spring is $\frac{1}{2}$ in. broad, $\frac{1}{50}$ in. thick, and 12 ft. long. What is the maximum torque it can exert on a central spindle if the stress is not to exceed 60,000 lb./in.^{2} ?

Problem 275. The links of a chain are made of $\frac{3}{4}$-in. round wrought iron, with semi-circular ends of radius 1 in. Straight portion of link 1 in. long. Find the maximum stresses in the link due to a pull of 1 ton on the chain.

Problem 276. A ring is made from a round steel rod 1 in. in diameter. The inside diameter of the ring is 6 in. Find the maximum stress resulting from a pull on the ring of $\frac{3}{4}$ ton.

Problem 277. Calculate the maximum tensile and compressive stresses on the cross section of the hydraulic riveter shown in Fig. 62, page 79.

Problem 278. In Fig. 133 a design is shown for the cross section of a punch press frame.* Substitute this cross section for that shown in Fig. 62, page 79, calculate the maximum stresses, and compare with the results of Problem 277.

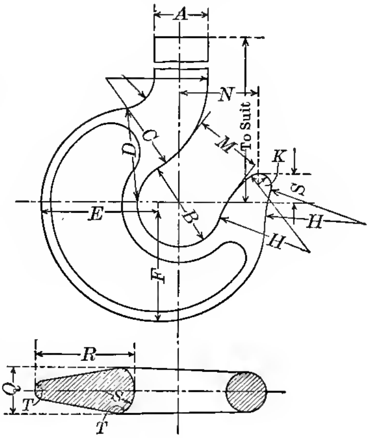


FIG. 134

Problem 279. The following table gives the dimensions of crane hooks for the design shown in Fig. 134 for loads from 5 to 50 tons.† Compute the maximum stresses at the dangerous section in each case and determine the factor of safety.

TONS	A	B	C	D	E	F	H	K	M	N	Q	R	S	T
5 .	$1\frac{7}{8}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	4	$3\frac{3}{4}$	$\frac{1}{2}$	$2\frac{3}{4}$	3	$1\frac{3}{4}$	$3\frac{1}{2}$	$1\frac{1}{4}$	$\frac{1}{2}$
10 . . .	$2\frac{5}{8}$	$4\frac{1}{2}$	5	5	6	$5\frac{3}{8}$	5	$\frac{5}{8}$	$3\frac{5}{8}$	4	$2\frac{1}{4}$	5	$1\frac{1}{2}$	$\frac{1}{2}$
15 . . .	3	$5\frac{1}{4}$	$6\frac{1}{4}$	6	$7\frac{1}{4}$	$6\frac{1}{4}$	$5\frac{3}{4}$	$\frac{3}{4}$	$4\frac{1}{4}$	$4\frac{5}{8}$	$2\frac{3}{4}$	6	$1\frac{3}{4}$	$\frac{5}{8}$
20 . . .	$3\frac{1}{2}$	6	$6\frac{1}{2}$	$6\frac{3}{4}$	$8\frac{1}{4}$	7	$6\frac{1}{2}$	$\frac{3}{4}$	$4\frac{3}{4}$	$5\frac{1}{4}$	$3\frac{1}{4}$	$6\frac{3}{4}$	2	$\frac{5}{8}$
25 . .	$3\frac{7}{8}$	$6\frac{3}{4}$	$7\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{4}$	8	$7\frac{3}{8}$	$\frac{3}{4}$	$5\frac{1}{2}$	6	$3\frac{3}{4}$	$7\frac{1}{2}$	$2\frac{1}{4}$	$\frac{3}{4}$
30 . . .	$4\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{1}{4}$	$8\frac{1}{4}$	$10\frac{1}{4}$	$8\frac{3}{4}$	$8\frac{1}{4}$	1	6	$6\frac{5}{8}$	4	$8\frac{1}{4}$	$2\frac{1}{2}$	$\frac{3}{4}$
40 . . .	$4\frac{3}{4}$	$8\frac{1}{2}$	$9\frac{1}{4}$	$9\frac{1}{2}$	$11\frac{3}{4}$	10	$9\frac{1}{4}$	1	$6\frac{3}{4}$	$7\frac{1}{2}$	$4\frac{1}{2}$	$9\frac{1}{2}$	3	1
50 . .	$5\frac{1}{4}$	$9\frac{1}{2}$	11	$10\frac{1}{2}$	12	$11\frac{1}{4}$	$10\frac{1}{4}$	1	$7\frac{1}{2}$	$8\frac{1}{4}$	5	$10\frac{1}{2}$	$3\frac{1}{2}$	$1\frac{1}{4}$

* Rautenstrauch, *Am. Mach.*, December 16, 1909.
† Dixon, *Am. Mach.*, August 16, 1900.

Problem 280. In Fig. 135 a design for the cross section of a crane hook is shown in which all the dimensions are expressed in terms of a single quantity r. The values of this constant r for various loads are given by the designer as follows:*

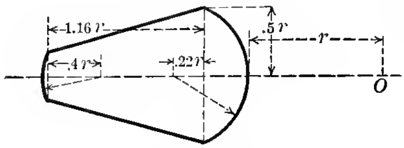


FIG. 135

40-ton hook, $r = 5.54$;
30-ton hook, $r = 4.7$;
20-ton hook, $r = 3.94$;
10-ton hook, $r = 2.76$;
5-ton hook, $r = 1.95$;
2-ton hook, $r = 1.23$.

Compare the strength of this design for a given load with that of the design shown in Fig. 134.

NOTE. Materials like cast iron, which do not conform to Hooke's law, cannot be subjected to a rigorous stress analysis. For example, the Pearson-Andrews formula is based on Poisson's ratio, which is one of the most refined elastic properties, and it is therefore useless to attempt to calculate the stress in a casting by such formulas. Moreover, it has recently been shown by experiment that the initial stresses due to cooling in an irregular casting, such as a punch or riveter frame, are so great as to upset any exact calculations of the bending stresses involved.† In these experiments many of the specimens failed by a vertical crack appearing in the web just back of the inner, or compression, flange, i.e. perpendicular to the section AB in Fig. 62, page 79, a form of failure which has no apparent relation to the theory of flexure. These experiments were also valuable in showing the practical necessity of putting a fillet in the corners where the web joins the inner flange, or increasing the thickness of the web at this point, as shown in Fig. 133.

In many machine tools the rigidity of the frame is the factor which determines the design, rather than the strength of the construction. In all such cases empirical methods based on practical experience are the ones that should be employed.

* Rautenstrauch, *Am. Mach.*, December 16, 1909.

† A. L. Jenkins, "The Strength of Punch and Riveter Frames made of Cast Iron," *Jour. Am. Soc. Mech. Eng.*, Vol. XXXII, pp. 311-332.

CHAPTER X

ARCHES AND ARCHED RIBS

I. Graphical Analysis of Forces

138. Composition of forces. In determining the effect which a given system of forces has upon a body, it is often convenient to represent the forces by directed lines and calculate the result graphically. In this method of representation the length of the line denotes the magnitude of the force laid off to any given scale, and the direction of the line indicates the direction in which the force acts, or its **line of action.**

When the lines of action of a system of forces all pass through the same point, the forces are said to be **concurrent.** The simplest method of dealing with such a system is to find the amount and line of action of a single force which would have the same effect as the given system of forces upon the motion of the point at which they act. This single force is called the **resultant** of the given system and its equal and opposite the **equilibrant.** When each of a system of forces acting on a body balances the others so that the body shows no tendency to move, the forces are said to be in **equilibrium,** in which case their resultant must be zero.

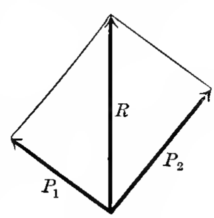


Fig. 136

The resultant of two forces acting at a point is found by drawing the forces to scale in both magnitude and direction, and constructing a parallelogram upon these two lines as adjacent sides; the diagonal of this parallelogram is then the required resultant (Fig. 136). This construction can be verified experimentally by fastening a string at two points A and B and suspending a weight R from it at any point C (Fig. 137). Then if two forces equal in magnitude to the tension in AC and BC are laid off parallel to AC and BC

respectively, it will be found that their resultant is equal and parallel to R, and opposite in direction.

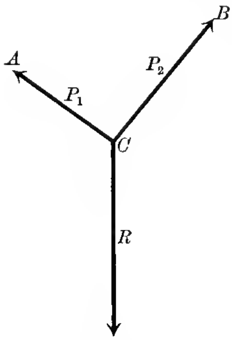


FIG. 137

Since the opposite sides of a parallelogram are equal and parallel, it is more convenient in finding the resultant of two forces to construct half the parallelogram. Thus, in the preceding example, if P_2 is laid off from the end of P_1, R is the closing side of the triangle so formed (Fig. 138). Such a figure is called a **force triangle.**

In order to find the resultant of several concurrent forces lying in the same plane, it is only necessary to combine two of them into a single resultant, combine this resultant with a third force, and so on, taking the forces in order around the point in which they meet. Thus, in Fig. 139, R_1 is the resultant of P_1 and P_2; R_2 is the resultant of R_1 and P_3; R_3 is the resultant of R_2 and P_4; and R is the resultant of R_3 and P_5. R is therefore the resultant of the entire system P_1, P_2, P_3, P_4, P_5.

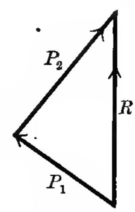


FIG. 138

In carrying out this construction it is unnecessary to draw the intermediate resultants R_1, R_2, and R_3, the final resultant in any case being the closing side of the polygon formed by placing the forces end to end in order. Such a figure is called a **force polygon.** From the above construction it is evident that the necessary and sufficient condition that a system of concurrent forces shall be in equilibrium is that their force polygon shall close, since in this case their resultant must be zero.

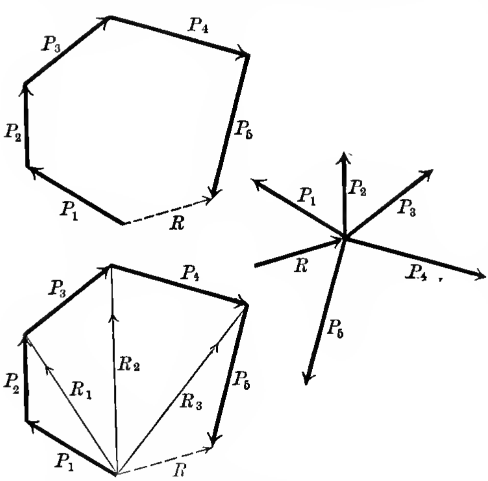


FIG. 139

The resultant of a system of non-concurrent forces lying in the same plane, that is to say, forces whose lines of action do not all pass through the same point, is found by means of a force polygon as explained above. In this case, however, the closing of the force polygon is not a sufficient condition for equilibrium, for the given system may reduce to a pair of equal and opposite forces acting in parallel directions, called a **couple**, which would tend to produce rotation of the body on which they act. For non-concurrent forces, therefore, the necessary and sufficient conditions for equilibrium are first, the resultant of the given system must be zero, and second, the sum of the moments of the forces about any point must be zero.

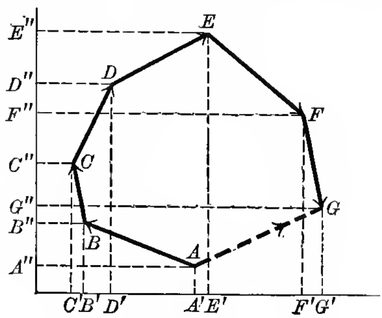


Fig. 140

Suppose that the force polygon corresponding to any given system of forces is projected upon two perpendicular lines, say a vertical and a horizontal line. Then since the sum of the projections upon any line of all the sides but one of a polygon is equal to the projection of this closing side upon the given line, the sum of the horizontal projections of any system of forces is equal to the horizontal projection of their resultant, and the sum of their vertical projections is equal to the vertical projection of their resultant (Fig. 140).

The conditions for equilibrium of a system of forces lying in the same plane may then be reduced to the following convenient form.

1. *For equilibrium against translation,*

$$\begin{cases} \sum \textbf{horizontal components} = 0, \\ \sum \textbf{vertical components} \quad = 0. \end{cases}$$

2. *For equilibrium against rotation,*

$$\sum \textbf{moments about any point} = 0.$$

If the forces are concurrent, rotation cannot occur, and the first condition alone is sufficient to assure equilibrium. In order that

a system of non-concurrent forces shall be in equilibrium, however, both conditions must be fulfilled.

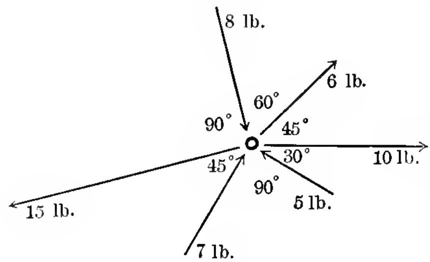


Fig. 141

Problem 281. Construct the resultant of the system of concurrent forces shown in Fig. 141.

Problem 282. Determine whether or not the system of parallel forces shown in Fig. 142 satisfies conditions 1 and 2 above.

139. Equilibrium polygon. The preceding construction for the force polygon gives a method for calculating the magnitude and direction of the resultant of any given system of forces, but does not determine the line of action of their resultant. The most convenient way to determine the line of action of the resultant is to introduce into the given system two equal and opposite forces of arbitrary amount and direction, such as P' and P'' in Fig. 143 (A).

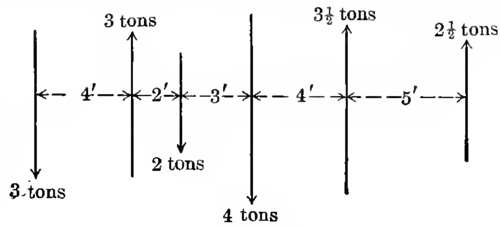


Fig. 142

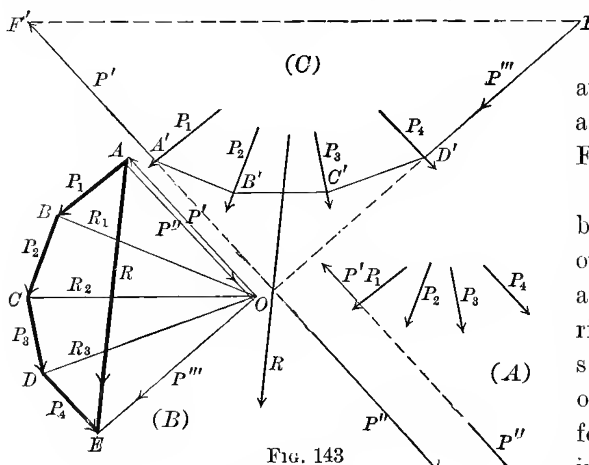


Fig. 143

Since P' and P'' balance one another, they will not affect the equilibrium of the given system. This is obvious from the force polygon. For in Fig. 143 (B), let R denote the resultant of the given system of forces $P_1 \cdots P_4$. Then, if OA represents in magnitude and direction the arbitrary force P', OB

is the resultant of P' and P_1, OC is the resultant of OB and P_2, etc., and finally OE, or P''', represents the resultant of P', P_1, P_2, P_3, P_4. If then P''' is combined with P'', the resultant R is obtained as before.

Now to find the line of action of R, suppose that P' and P_1 are combined into a resultant R_1 acting in the direction $B'A'$ (Fig. 143 (C)) parallel to the ray OB of the force polygon (Fig. 143 (B)). Prolong $A'B'$ until it intersects P_2, and then combine R_1 and P_2 into a resultant R_2 acting in the direction $C'B'$ parallel to the ray OC of the force polygon. Continue in this manner until P''' is obtained. Then the resultant of P'' and P''' will give both the magnitude and line of

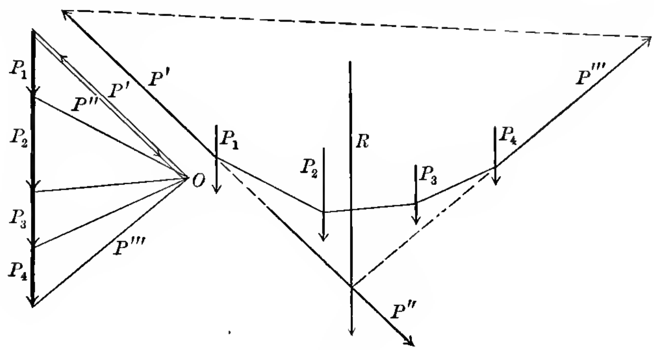


Fig. 144

action of the resultant of the original system P_1, P_2, P_3, P_4. The closed figure $A'B'C'D'E'F'$ obtained in this way is called an **equilibrium polygon.**

For a system of parallel forces the equilibrium polygon is constructed in the same manner as above, the only difference being that in this case the force polygon becomes a straight line, as shown in Fig. 144.

Since P' and P'' are entirely arbitrary both in magnitude and direction, the point O, called the **pole**, may be chosen anywhere in the plane. Therefore, in constructing an equilibrium polygon corresponding to any given system of forces, the force polygon $ABCDE$ (Fig. 143) is first drawn, then any convenient point O is chosen and joined to the vertices A, B, C, D, E of the force polygon, and finally the equilibrium polygon is constructed by drawing its sides parallel to the rays OA, OB, OC, etc., of the force diagram.

Since the position of the pole O is entirely arbitrary, there is an infinite number of equilibrium polygons corresponding to any given set of forces. The position and magnitude of the resultant R, however, is independent of the choice of the pole, and will be the same, no matter where O is placed.

Problem 283. The ends of a cord are fastened to supports and weights attached at different points of its length. Show that the position assumed by the string is the equilibrium polygon for the given system of loads.

140. Application of equilibrium polygon to determining reactions. One of the principal applications of the equilibrium polygon is in determining the unknown reactions of a beam or truss. To illustrate its use for this purpose, consider a simple beam placed horizontally and bearing a number of vertical loads P_1, P_2, etc. (Fig. 145). To determine the reactions R_1 and R_2, the force diagram is first

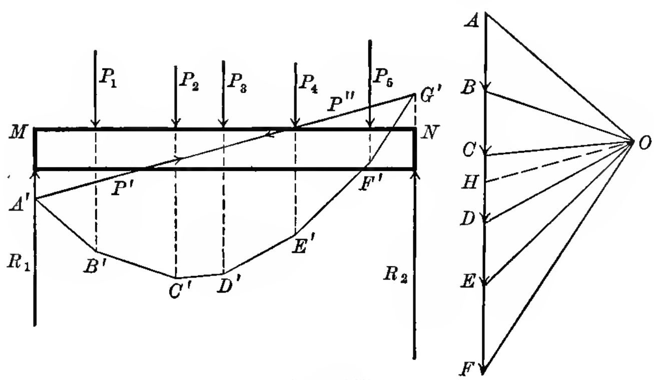


FIG. 145

constructed by laying off the loads P_1, P_2, etc., to scale on a line AF, choosing any convenient point O as pole and drawing the rays OA, OB, etc. The equilibrium polygon corresponding to this force diagram is then constructed, starting from any point, say A', in R_1.

Now the closing side $A'G'$ of the equilibrium polygon determines the line of action of the resultants P' and P'' at A' and G' respectively. For a simple beam, however, the reactions are vertical. Therefore, in order to find these reactions each of the forces P' and P'' must be resolved into two components, one of which shall be vertical. To accomplish this, suppose that a line OH is drawn from the pole O in

the force diagram parallel to the closing side $G'A'$ of the equilibrium polygon. Then HO (or P') may be replaced by its components HA and AO, parallel to R_1 and $A'B'$ respectively; and similarly, OH may be replaced by its components FH and OF, parallel to R_2 and $F'G'$ respectively. HA and FH are therefore the required reactions.

Problem 284. A simple beam 20 ft. long supports concentrated loads of 3, 5, 2, and 9 tons at distances of 5, 7, 14, and 18 ft. respectively from the left support. Calculate the reactions of the supports graphically.

Problem 285. Construct an equilibrium polygou for a simple beam bearing a uniform load, and show that the reactions are equal.

141. Equilibrium polygon through two given points. Let it be required to pass an equilibrium polygon through two given points, say M and N (Fig. 146).

To solve this problem a trial force diagram is first drawn with any arbitrary point O as pole, and the corresponding equilibrium polygon

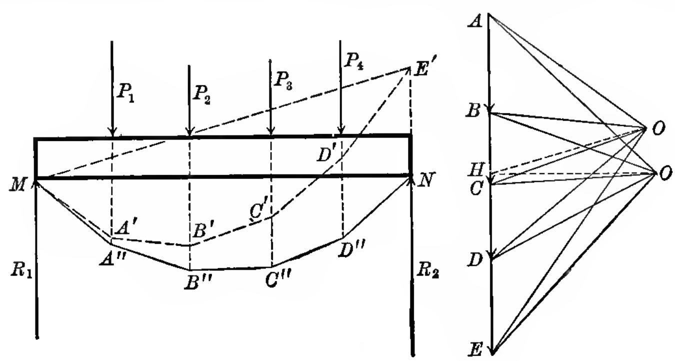


FIG. 146

$MA'B'C'D'E'$ constructed, starting from one of the given points, say M. The reactions are then determined by drawing a line OH parallel to the closing side ME' of the equilibrium polygon, as explained in the preceding article.

The reactions, however, are independent of the choice of the pole in the force diagram, and consequently they must be of amount AH and HE, no matter where O is placed. Moreover, if the equilibrium polygon is to pass through both M and N, its closing side must coincide with the line MN, and therefore the pole of the force diagram must lie somewhere on a line through H parallel to MN. Let O' be

a point on this line. Then if a new force diagram is drawn with O' as pole, the corresponding equilibrium polygon starting at M will pass through N.

142. Equilibrium polygon through three given points. Let it be required to pass an equilibrium polygon through three given points, say M, N, and L (Fig. 147).

As in the preceding article, a trial force diagram is first drawn with any point O as pole, and the corresponding equilibrium polygon constructed, thus determining the reactions R_1 and R_2 as AH and HE respectively.

Now if the equilibrium polygon is to pass through N, the pole of the force diagram must lie somewhere on a line HK drawn through

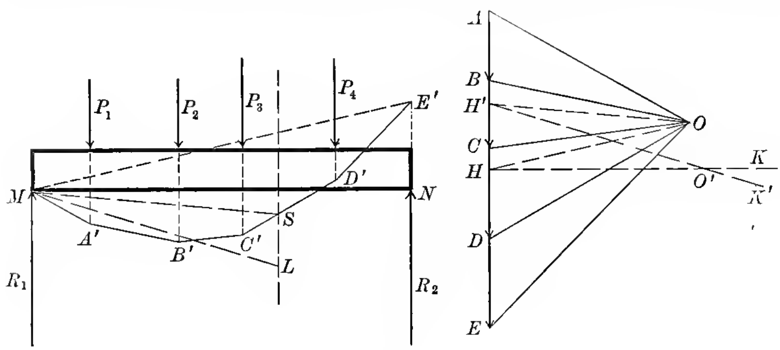


FIG. 147

H parallel to MN, as explained in the preceding article. The next step, therefore, is to determine the position of the pole on this line HK, so that the equilibrium polygon through M and N shall also pass through L. This is done by drawing a vertical LS through L and treating the points M and L exactly as M and N were treated. Thus $OABCD$ is the force diagram for this portion of the original figure, and $MA'B'C'S$ is the corresponding equilibrium polygon, the reactions for this partial figure being $H'A$ and DH'. If, then, the equilibrium polygon is to pass through L, its closing side must be the line ML, and consequently the pole of the force diagram must lie on a line $H'K'$ drawn through H' parallel to ML. The pole is therefore completely determined as the intersection O' of the lines HK and $H'K'$. If, then, a new force diagram is drawn with O' as pole, the

corresponding equilibrium polygon starting from the point M will pass through both the points L and N.

Since there is only one position of the pole O', but one equilibrium polygon can be drawn through three given points. In other words, an equilibrium polygon is completely determined by three conditions.

143. Application of equilibrium polygon to calculation of stresses. Consider any structure, such as an arch or arched rib, supporting a system of vertical loads, and suppose that the force diagram and equilibrium polygon are drawn as shown in Fig. 148. Then each ray of the force diagram is the resultant of all the forces which precede it, and acts along the segment of the equilibrium polygon parallel to this ray. For instance, OC is the resultant of all the forces on the

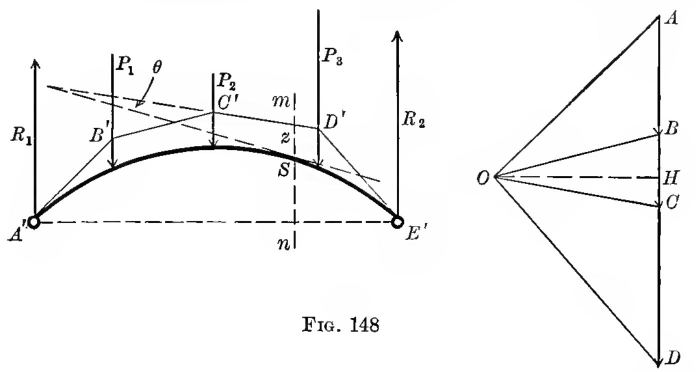


FIG. 148

left of P_3, and acts along $C'D'$. Consequently the stresses acting on any section of the structure, say mn, are the same as would result from a single force OC acting along $C'D'$.

Let θ denote the angle between the segment $C'D'$ of the equilibrium polygon and the tangent to the arch at the point S. Then the stresses acting on the section mn at S are due to a tangential thrust of amount $OC \cos \theta$; a shear at right angles to this, of amount $OC \sin \theta$; and a moment of amount $OC \cdot d$, where d is the perpendicular distance of $C'D'$ from S.

From Fig. 148, it is evident that the horizontal component of any ray of the force diagram is equal to the pole distance OH. Therefore if OC is resolved into its vertical and horizontal components, the moment of the vertical component about S is zero, since it passes

through this point; and hence the moment $OC \cdot d = OH \cdot z$, where z is the vertical intercept from the equilibrium polygon to the center of moments S. Having determined the moment at any given point, the stresses at this point can be calculated as explained in Article 157.

144. Relation of equilibrium polygon to bending moment diagram. In the preceding article it was proved that the moment acting at any point of a structure is equal to the pole distance of the force diagram multiplied by the vertical intercept on the equilibrium polygon from the center of moments. For a system of vertical loads, however, the pole distance is a constant. Consequently the moment acting on any section is proportional to the vertical intercept on the equilibrium polygon from the center of moments. Therefore, if the equilibrium polygon is drawn to such a scale as to make this factor of proportionality equal to unity, the equilibrium polygon will be identical with the bending moment diagram for the given system of loads.

Problem 286. Compare the bending moment diagrams and equilibrium polygons for the various cases of loading illustrated in Article 52.

II. Concrete and Masonry Arches

145. Definitions and construction of arches. The following discussion of the arch applies only to that form known as the *barrel arch.* Domed and cloistered arches demand a special treatment which is beyond the scope of this volume.

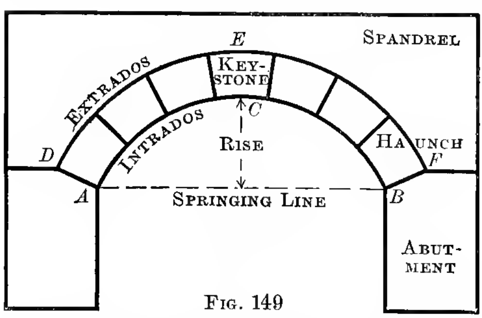


Fig. 149

The various portions of a simple, or barrel, arch, such as shown in projection in Fig. 149, have the following special names.

Soffit: the inner or concave surface of the arch.

Intrados: the curve of intersection (ACB, Fig. 149) of the soffit, with a vertical plane perpendicular to the axis, or length, of the arch.

Extrados: the curve of intersection (DEF, Fig. 149) of a vertical plane with the outer surface of the arch.

Crown: the highest part of the arch.

Haunches: the parts of the arch next to the abutments.

Springing line: the line AB joining the ends of the intrados.

Rise: the distance from the springing line to the highest point of the intrados.

Spandrel: the space above the extrados. In the case of an arch supporting a roadway, the filling deposited in this space is called the *spandrel filling.*

Voussoir: any one of the successive stones in the arch ring of a masonry arch.

Keystone: the central voussoir.

In constructing an arch the material is supported while being put in place by a wooden structure called a *center*, the outer surface of which has the exact form of the soffit of the required arch. The center is constructed by making a number of frames or ribs having the form of the intrados of the required arch, and then placing these ribs at equal intervals along the axis of the arch and covering them with narrow wooden planks, called *lagging,* running parallel to the axis of the arch. When the arch is completed, or, in case of a concrete arch, when the material has hardened sufficiently to resist the stress due to its weight, the centers are removed, thus leaving the arch self-supporting.

146. Load line. Since the filling above an arch has the same form as the arch itself, it must be partly self-supporting. In designing an arch, however, no advantage is taken of this fact, and it is assumed that any portion of the extrados supports the entire weight of the material vertically above it. The only exception to this is in the construction of tunnel walls, in which case it would be obviously unnecessary as well as impracticable to construct an arch sufficiently strong to support the entire weight of the material above it.

If the filling above an arch is not of the same material as the arch ring, subsequent calculations are greatly simplified by constructing a **load line** which shall represent at any point the height which a filling of the same material as the arch itself must have in order to produce the same load as that actually resting on the arch. The vertical intercept between the intrados and the load line will then represent the load at any given point of the arch.

In case of a live load the load line will have a different form for each position of the moving load.

Problem 287. A circular arch of 20 ft. span and 6 ft. rise, with an arch ring 3 ft. thick, is composed of concrete weighing 140 lb./ft.^{3} Construct the load line

for a roadway three feet above the crown of the arch, with a spandrel filling of earth weighing 100 lb./ft.^{3}

Solution. In this case the weight of a cubic foot of the spandrel filling is to the weight of a cubic foot of the arch ring as 100 : 140. Therefore the load line is obtained by reducing the intercept on each ordinate between the roadway and the extrados in the ratio 140 : 100. Thus, in Fig. 150, reducing any ordinate AB in this ratio we obtain the ordinate BC, etc. By carrying out this reduction on a sufficient number of ordinates, and joining the points C so found, the load line $DECFG$ is obtained.

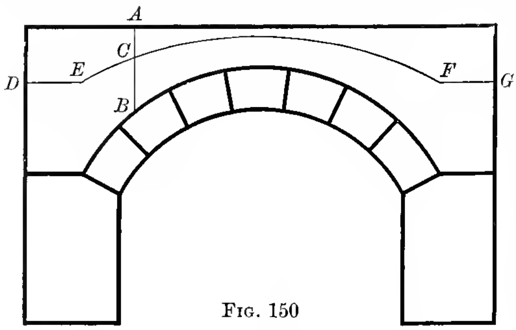


Fig. 150

147. Linear arch. Suppose that the voussoirs of an arch have slightly curved surfaces so that they can rock on one another, as shown in Fig. 151. The points of contact of successive voussoirs are then called centers of pressure, and the line joining them the line of pressure, or **linear arch.** It is evident, from the figure, or from a model constructed as above, that with every change of loading the voussoirs change their position more or less, thus altering the form of the linear arch. In a model constructed as above, the linear arch can alter its shape considerably without overthrowing the structure, the only condition necessary to assure stability being that the linear arch shall lie within the middle third.*

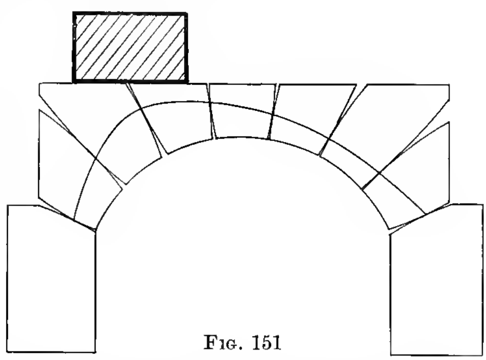


Fig. 151

In a masonry arch the pressure on any joint is ordinarily distributed over the entire surfaces in contact. In this case the center of pressure is the point of application of the resultant joint pressure, and

* See discussion of arches in article by Fleeming Jenkin, entitled "Bridges," *Encyclopædia Britannica*, 9th ed., Vol. IV, pp. 273–282.

the linear arch is the broken line joining these centers of pressure. In a concrete arch the linear arch becomes a continuous curve. With each change of loading the same shifting of the linear arch occurs as in the case of the model with curved joints, the only difference being that with flat joints this action is not visible. To assure stability, however, the linear arch must be restricted to lie within the middle third of the arch ring, as will be proved in Article 148.

If we consider a single voussoir of a masonry arch, or a portion of a concrete arch bounded by two plane sections, as shown in Fig. 152, the resultant joint pressures R and R', and the weight P of the block and the material directly above it, form a system of forces in equilibrium. Consequently, if the amount, direction, and point of application of one of these resultant joint pressures are known, the amount, direction, and point of application of the other can be found by constructing a triangle of forces. Therefore, if one resultant joint pressure is completely known in position, amount, and direction, the others can be successively found as above, thus determining the linear arch as an equilibrium polygon for the given system of loads.

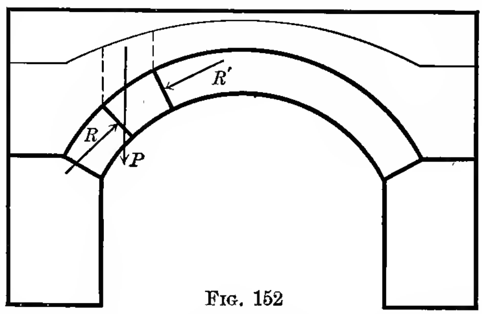


Fig. 152

Since an equilibrium polygon may be drawn to any given scale, if no one joint pressure is completely known, which is usually the case, there will be, in general, an infinite number of equilibrium polygons corresponding to any given system of loads. The linear arch may, however, be defined as that particular equilibrium polygon which coincides with the pressure line, and the question then arises how to determine the equilibrium polygon so that it shall coincide with the pressure line. This problem will be discussed more fully in Articles 150, 151, and 152.

When the linear arch has been determined, the resultant pressure on a joint having any inclination to the vertical can easily be obtained. Thus, in Fig. 153, let R be the resultant pressure on a

vertical section through B, and R' the resultant pressure on the inclined section AE through B. Since R is due to the load on the right of the vertical CF, and R' to the load on the right of the broken line DAE, the difference between them must be due to the load $ABCD$ minus the load BFE. Let P denote the difference between these two loads, represented by the shaded portion in Fig. 153. Then, since R, R', and P must be in equilibrium, R' is found at once by drawing a force triangle, as shown in the figure.

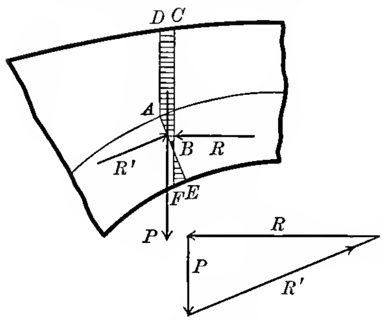


FIG. 153

148. Conditions for stability. A masonry arch may fail in any one of three ways: (1) by sliding of one voussoir upon another; (2) by overturning; (3) by crushing of the material.

These three methods of failure will now be considered in order.

1. The first method of failure is caused by the shearing stress at any joint exceeding the joint friction, or the adhesion of the mortar. This kind of failure can only occur when the angle which the resultant pressure on any joint makes with a normal to the plane of the joint exceeds the angle of repose for the material in question (Article 167). Ordinarily the resultant pressure on any joint is very nearly perpendicular to its plane, and since the angle of repose for masonry is very large, failure by sliding is not likely to occur.

As a criterion for safety against failure of this kind, it may be assumed that when the resultant makes an angle of less than 30° with the normal to the joint safety against sliding is assured.

2. In order for an arch to fail by overturning, one or more of the joints must open at one edge, the adjacent blocks rotating about their center of pressure. For this to occur, one edge of the joint must be in tension. Although in a well-laid masonry arch the joints have considerable tensile strength, it is customary to disregard this entirely, and in this case the condition necessary to assure stability against rotation is that every joint shall be subjected to compressive stress only. Assuming, then, a linear distribution of

stress over the joints, the center of pressure is restricted to lie within the middle third of any joint (compare Article 62).

Thus, in Fig. 154 (A), if $ABCD$ represents the distribution of pressure on any joint AD, the resultant R must pass through the center of gravity of the trapezoid $ABCD$. Consequently, when the compression at one edge becomes zero, as shown in Fig. 154 (B), the resultant R is applied at a point distant $\frac{b}{3}$ from A, and cannot approach any nearer to A without producing tensile stress at D. Therefore, the criterion for stability against overturning is that the center of pressure on any joint shall not approach nearer to either edge than $\frac{b}{3}$, where b is the width of the joint; or, in other words, that *the linear arch must lie within the middle third of the arch ring.*

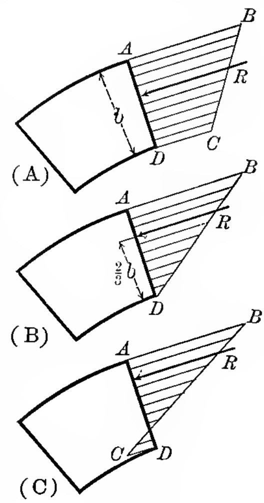


FIG. 154

3. Failure by crushing can only occur when the maximum stress on any joint exceeds the ultimate compressive strength of the material. To guard against this kind of failure, 10 is universally chosen as the factor of safety. Hence, if u_c denotes the ultimate compressive strength of the material, and $p_{\max}$ the maximum unit stress on any joint, the criterion for stability against crushing is

$$p_{\max} < \frac{u_c}{10}.$$

From Fig. 154 (B), the maximum unit stress is twice the average. Therefore, if F denotes the area of a joint, and p_a the average unit stress on it,

$$p_a = \frac{R}{F} \quad \text{and} \quad p_{\max} = 2\,p_a.$$

Consequently the criterion for stability against crushing can be expressed in the more convenient form

$$\frac{R}{F} < \frac{u_c}{20};$$

that is to say, *the average unit stress on any joint must not exceed one twentieth of the ultimate compressive strength of the material.*

The above conditions for stability can be applied equally as well to a concrete arch by considering the stress on any plane section of the arch ring.

149. Maximum compressive stress. Let R denote the resultant pressure on any joint, b the width of the joint, F its area, and c the distance of the center of pressure from the center of gravity of the joint. Then, under the assumption of a linear distribution of stress, the stress on the joint is due to a uniformly distributed thrust of amount $\frac{R}{F}$ per unit of area, and a moment M of amount $M = Rc$. Therefore the unit stress p at any point is given by the formula

$$p = \frac{R}{F} \pm \frac{Me}{I},$$

where e is the distance of the extreme fiber from the center of gravity, and I is the moment of inertia of the cross section.

For a section of unit length, $F = b \cdot 1 = b$, $I = \frac{b^3}{12}$, and $e = \frac{b}{2}$. Therefore, substituting these values, the formula for maximum or minimum stress becomes

$$p_{\substack{\max \\ \min}} = \frac{R}{b} \pm \frac{6\,Rc}{b^2}.$$

For $c = \frac{b}{6}$ the minimum stress is zero, and if $c > \frac{b}{6}$ it becomes negative, thus restricting the center of pressure to lie within the middle third of the cross section if tensile stress is prohibited (compare Article 62 and Article 148, 2).

Combining this result with that of the preceding article, the maximum stress calculated by the formula

$$p_{\max} = \frac{R}{b} + \frac{6\,Rc}{b^2}$$

must not exceed $\frac{u_c}{10}$, where u_c is the ultimate compressive strength of the material.

150. Location of the linear arch: Moseley's theory. In order to obtain a starting point for the construction of the linear arch, it is necessary to know the amount, direction, and point of application

of one joint pressure, as explained in Article 147; or, in general, it is necessary to have given three conditions which the equilibrium polygon must satisfy, such, for instance, as three points through which it is required to pass. Since it is impossible to determine these three unknowns by the principles of mechanics, the theory of the arch has long been a subject of controversy among engineers and mathematicians.

Among the various theories of the arch which have been proposed from time to time, the first and most important of the older theories is called the **principle of least resistance.** This theory was introduced by the English engineer, **Moseley,** in 1837, and later became famous on the Continent through a German translation of Moseley's work by **Scheffler.**

In building an arch the material is assembled upon a wooden framework called a center; when the arch is complete this center is removed and the arch becomes self-supporting, as explained in Article 145. Now suppose that instead of removing the center suddenly, it is gradually lowered so that the arch becomes self-supporting by degrees. In this case the horizontal pressure or thrust at the crown gradually increases until the center has been completely removed, when it has its least possible value. This hypothesis of least crown thrust consistent with stability is Moseley's principle of least resistance.

In constructing an equilibrium polygon the horizontal force, or pole distance, is least when the height of the polygon is a maximum. Therefore, in order to apply the principle of least resistance, the equilibrium polygon must pass through the highest point of the extrados at the crown and the lowest points of the intrados at the abutments. Since this would cause tensile stress at both the crown and abutments, the criterion for stability against overturning makes it necessary in applying the theory to move the center and ends of the equilibrium polygon, or linear arch, until it falls within the middle third of the arch ring. There is nothing in the principle of least resistance, however, to warrant this change in the position of the equilibrium polygon, and consequently the theory is inconsistent with its application.

Culmann tried to overcome this objection to Moseley's theory by considering the compressibility of the mortar between the joints. At the points of greatest pressure the mortar will be compressed more

than elsewhere, and this will cause the pressure line, or linear arch, to move down somewhat, thus taking a position nearer to the middle third than is required by the principle of least resistance, if applied to the arch as a rigid body.

The above brief account of Moseley's principle of least resistance and Culmann's modification of it are given chiefly for their historical interest and the importance formerly attached to them. The modern theory of the arch is based upon the principle of least work, and is therefore rigorously correct from the standpoint of the mathematical theory of elasticity.

151. Application of the principle of least work. Although Hooke's law is not rigorously true for such materials as stone, cement, and concrete, the best approximation to actual results is obtained by assuming that the materials of which the arch is composed conform to Hooke's law, and then basing the theory of the arch on the general theorems of the strength of materials. On this assumption the position of the linear arch can be determined by means of Castigliano's theorem, which states that for stable equilibrium the work of deformation must be a minimum (Articles 79 and 81).

Consider a section of the arch perpendicular to the center line of the arch ring, or, in general, normal to the intrados. Let F denote the area of the section, R the resultant pressure on the section, c the distance of the point of application of R from the center of gravity of the section, and ds an infinitesimal element of the center line. Then the work of deformation will consist of two parts, — that due to the axial thrust R, and that due to a moment $M = Rc$. Since the direct stress per unit of area of the section is $\frac{R}{F}$, the unit deformation due to the stress is $\frac{R}{FE}$, where E denotes Young's modulus; and hence the work of deformation due to R is $\frac{1}{2}R\left(\frac{R}{FE}\right)$, or $\frac{R^2}{2\,FE}$. From Article 73, Chapter IV, the work of deformation due to the bending moment M is $\frac{M^2}{2\,EI}$. Therefore the work of deformation dW for a portion of the arch included between two cross sections at a distance ds apart is

$$dW = \frac{R^2}{2\,EF}\,ds + \frac{M^2}{2\,EI}\,ds.$$

Hence the total work of deformation for the entire arch is

$$W = \int \frac{R^2}{2\,EF}\,ds + \int \frac{M^2}{2\,EI}\,ds.$$

Let b denote the thickness of the arch ring, and consider a section of unit width. Then $F = b$ and $I = \frac{b^3}{12}$, and substituting these values in the above equation and assuming that E is constant throughout the arch,

$$W = \frac{1}{2\,E}\int_0^s \left(\frac{R^2}{b} + \frac{12\,M^2}{b^3}\right) ds.$$

In Article **147** it was shown that three conditions are necessary for the determination of the linear arch. Therefore, since the values of R and M in the above expression depend upon the position of the linear arch, in order to apply Castigliano's theorem to the integral, R and M must first be expressed in terms of these three unknown quantities, which may be conveniently chosen as the position, amount, and direction of the joint pressure at a certain point.

Having expressed R and M in this way, Castigliano's theorem is applied by differentiating W partially with respect to each of the three unknowns, and equating these three partial derivatives to zero. In this way three simultaneous equations are obtained which may be solved for the three unknown quantities, thus completely determining the linear arch.

The principle of least work, therefore, permits of a rigorously correct determination of the linear arch. Instead, however, of actually carrying out the process outlined above, Winkler has applied the principle to the derivation of a simple criterion for stability, as explained in the following article.

152. Winkler's criterion for stability. From the preceding article, the total work of deformation for the whole arch is given by the expression

$$W = \frac{1}{2\,E}\int_0^s \left(\frac{R^2}{b} + \frac{12\,M^2}{b^3}\right) ds,$$

in which the integral is to be extended over the entire length of the arch. As the position of the pressure line is altered, the first term in this integral changes but little, whereas the second term undergoes a considerable variation, since $M = Rc$, where c is the distance

of the center of pressure from the center of gravity of the section. For a first approximation, therefore, the first term may be disregarded in comparison with the second, and hence the problem of making W a minimum reduces to that of making the integral $\int \frac{M^2}{b^3} ds$ as small as possible.

To effect a still further reduction, suppose that R is resolved into vertical and horizontal components so that the vertical component shall pass through the center of gravity G of the section (Fig. 155), and let z denote the perpendicular distance of the horizontal component P_h from G. Then $M = P_h z$ and the integral $\int \frac{M^2}{b^3} ds$ becomes $\int \frac{P_h^2 z^2}{b^3} ds$, or, since P_h is constant for all sections, this may be written $P_h^2 \int \frac{z^2 ds}{b^3}$.

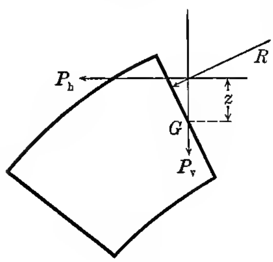


FIG. 155

Ordinarily the thickness of the arch ring varies, being least at the crown and greatest at the abutments. In this case let b_c denote the thickness of the crown, and suppose that the law of variation in thickness is such that the thickness b at any other point is given by the expression

$$b^3 = b_c^3 \frac{ds}{dx},$$

where dx is the horizontal projection of ds. Under this assumption, the expression $P_h^2 \int \frac{z^2 ds}{b^3}$ becomes

$$\frac{P_h^2}{b_c^3} \int z^2 dx.$$

Therefore the problem of making W a minimum is now reduced to that of making the integral $\int z^2 dx$ as small as possible.

This latter expression, however, consists of only positive terms, and reduces to zero for the center line of the arch. From this it follows that if an equilibrium polygon is drawn for the given system of loads, and then the center line of the arch is so chosen as to coincide with this equilibrium polygon, the true linear arch can differ but little from this center line.

In order for an arch to be stable at least one of the many possible assumptions of the linear arch must be such as to fall within the middle third of the arch ring. Moreover, the elastic deformation of the arch is such as to move the linear arch as near to the center line as the form of the arch permits. Therefore, *if for any given arch it is possible to draw an equilibrium polygon which shall everywhere lie within the middle third of the arch ring, the stability of the arch is assured.*

This criterion for stability is due to Winkler, and was first given by him in 1879.

153. Empirical formulas. The thickness necessary to give an arch at the crown can only be found by assuming a certain thickness and determining whether or not this satisfies all the conditions of stability. The least thickness consistent with stability is such that the average compressive stress does not exceed one twentieth of the ultimate compressive strength of the material. The arch is usually made somewhat thicker than is required by this criterion, however, for the thicker the arch the more easily can the equilibrium polygon be made to lie within the middle third of the arch ring.

The following empirical formulas for thickness at crown represent the best American, English, and French practice respectively, and may be used in making a first assumption as a basis for calculations.

$$b = \frac{\sqrt{r + \frac{l}{2}}}{4} + 0.2; \qquad \textbf{Trautwine.}$$

$$b = \sqrt{.12\,r}; \qquad \textbf{Rankine.}$$

$$b = 1\frac{1}{12} + \frac{l}{23}; \qquad \textbf{Perronnet.}$$

r = radius of intrados in feet; d = rise in feet;

l = span in feet; b = depth at crown in feet.

154. Designing of arches. In designing an arch to support a given loading the equilibrium polygon for the given system of loads should, in accordance with Winkler's criterion, be assumed as the center line of the arch. This, however, is not always possible. For instance, in

the case of an arch intended to support a roadway, the level of which is fixed, the loading depends to a large extent on the form of the arch, and consequently the equilibrium polygon cannot be determined until the form of the arch has been assumed.

In designing arches, therefore, the method usually followed is to assume the form of the intrados of the required arch, and determine its thickness at the crown by an empirical formula, such as those given in the preceding article. Then, having drawn the extrados and load line, the surface between the intrados and the load line is divided into any convenient number of parts by drawing verticals, and the amount and position of the resultant weight of each part for a section one foot wide is calculated. An equilibrium polygon for this system of loads is then passed through the middle point of the arch ring at crown and abutments by the method given in Article 142. If this equilibrium polygon lies within the middle third of the arch ring, the arch is assumed to be stable against overturning.

If the equilibrium polygon through the middle points of the arch ring at crown and abutments does not lie entirely within the middle third of the arch ring, these three points are shifted so as to make it do so if possible. If no choice of the three points will make the equilibrium polygon lie entirely within the middle third of the arch ring, the design must be altered until this has been accomplished.

The next step is to calculate the maximum unit joint pressure by the formula given in Article 149, and apply the criterion for stability against crushing given in Article 148. When these criteria have been satisfied the design is assumed to be safe. If, however, there is a considerable excess of strength, the design may be lightened and the criteria reapplied.

Before the design can be considered complete it must also be shown that the above criteria are satisfied for every form of loading to which the arch is likely to be subjected. In the case of an arch designed to carry a heavy live load, such as that due to several locomotives, it may be necessary to draw a number of load lines corresponding to different positions of the load, and make a corresponding number of determinations of the equilibrium polygon and maximum joint pressure.

The stability of the abutments still remains to be investigated, and finally the bearing power of the soil on which these abutments rest.

Problem 288. Design a concrete arch to span a stream 25 ft. in width and support a roadway 15 ft. above the level of the stream, if the spandrel filling is clay weighing 120 lb./ft.^{3}; the maximum depth of frost is $3\frac{1}{2}$ ft. and the bearing power of the soil at this depth is 4 tons/ft.^{2} (see Article 158).

155. Stability of abutments. To determine the stability of the abutments, the joint pressure at the hauuch is combined with the weight of the abutment into a single resultant, say R'. For stability against overturning, the line of action of this resultant must strike within the middle third of the base (Article 148, 2).

Resolving the resultant R' into a horizontal component R_h and a vertical component R_v, the maximum pressure on the soil is calculated by substituting this value of R_v for R in the formula given in Article 149. To prevent sinking of the abutments, this pressure must not exceed the bearing power of the soil (see Article 166).

For stability against sliding, the shearing stress between the abutment and the soil, due to the horizontal component R_h of the resultant R', must be less than the friction between the two; or, more briefly, the angle which R' makes with the horizontal must be less than the angle of repose (compare Article 172).

156. Oblique projection of arch. Suppose that an arch, its load line, and its pressure line are drawn to any given scale, and then the whole figure is projected upon an oblique plane by a system of parallel lines. The projection of the pressure line on this oblique plane will then be the true pressure line for the projected arch and its projected load line.

This principle can often be used to advantage, as, for example, in comparing two arches of equal span but different rise. Its most important application is in giving an accurate construction of the pressure line for arches of long span and small rise. Thus, instead of plotting such an arch to scale, its projection can be plotted; or, in other words, its span can be shortened any convenient amount. A larger unit can then be used in plotting the vertical dimensions than would otherwise be possible, and consequently the pressure line can be drawn to any desired degree of accuracy.

Having constructed the pressure line in this way, the pressure on any joint of the given arch can be found from the pressure on the

corresponding joint of the projected arch by laying off the horizontal and vertical components of the latter to two different scales; in other words, by projecting the pressure back again onto the original arch.

III. Arched Ribs

***157. Stress in arched ribs.** The arch is frequently used in metal constructions, especially in such structures as roofs and bridges, in the form of a curved beam composed either of a solid web and flanges or built up like a truss. Such a metal arch is called an **arched rib.**

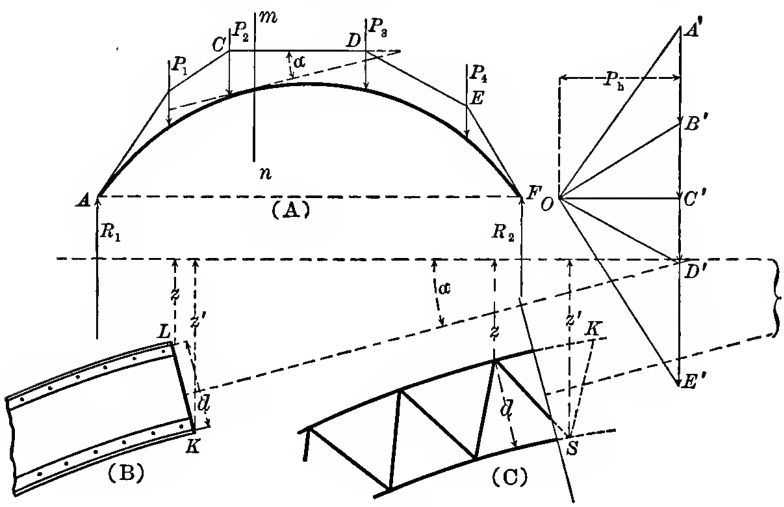


FIG. 156

The fundamental difference between a concrete or masonry arch and an arched rib is that the latter, being composed of metal, is capable of resisting bending moment. For an arched rib, therefore, it is not essential that the equilibrium polygon shall lie within the boundaries of the arch; it may, in fact, either cross the arch or lie entirely on either side, the only condition for stability being that the arched rib must be sufficiently strong to resist the bending moment thus produced.

* For a brief course the remainder of this chapter may be omitted.

When the equilibrium polygon has been drawn for the given system of loads, the stress at any point of an arched rib can be calculated by the method explained in Article 143. Thus, in Fig. 156 (A), let AGF denote the arched rib, P_1, P_2, etc. the given loads, and $ABCDEF$ the corresponding equilibrium polygon. Then the stress on any section mn is due to a force acting in the direction CD, of amount equal to the corresponding ray OC' of the force diagram.

Consequently, if the rib is composed of a solid web and flanges, as shown in Fig. 156 (B), the direct stress on the section is equal in amount to the ray OC' of the force diagram, the bending stress on the upper flange is $\frac{P_h z'}{d}$, the bending stress on the lower flange is $\frac{P_h z}{d}$, and the shear normal to the rib is $OC' \sin \alpha$, where α is the angle between CD and the tangent to the rib at the section.

Similarly, for the trussed rib shown in Fig. 156 (C), by taking moments about L and S the stresses in RS and LK are found to be $\frac{P_h z}{d}$ and $\frac{P_h z'}{d}$ respectively, while the normal component of the stress in LS is $OC \sin \alpha$.

Arched ribs are usually constructed in one of three different ways: (1) hinged at the abutments and at the crown; (2) hinged at the abutments and continuous throughout; (3) fixed at the abutments and continuous throughout. The method of constructing the equilibrium polygon differs for each of these three methods of support, and will be treated separately in what follows.

158. Three-hinged arched rib. When a member is free to turn at any point the bending moment at that point is zero, and consequently the equilibrium polygon, or bending moment diagram, passes through the point. For a three-hinged arched rib, therefore, the equilibrium polygon must pass through the centers of the three hinges and is therefore completely determined, as explained in Article 142.

159. Two-hinged arched rib. Consider an arched rib hinged at the ends and continuous between these points. In this case the equilibrium polygon must pass through the centers of both hinges, but since there is no restriction on the vertical scale, this scale may be anything whatever, depending on the choice of the pole in the

force diagram. A third condition is therefore necessary in order to make the problem determinate.

The problem can be solved in various ways, depending on the choice of the third condition. The first solution that will be given is that found by applying the principle of least work, that is, by applying Castigliano's condition that the work of deformation shall be a minimum.

Consider a two-hinged arched rib supporting a system of vertical loads, as shown in Fig. 157. Then the moment at any point A is equal to the moment of the forces on the left of the section mn through A, minus the moment of P_h about A, where P_h is the unknown

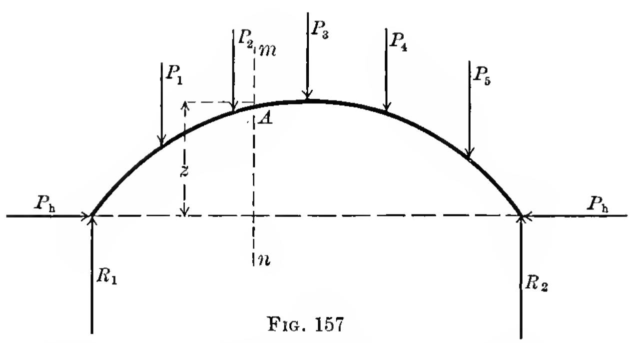


FIG. 157

horizontal reaction, or pole distance of the force diagram, which is to be determined. Consequently, if M denotes the moment at A, M_p the moment of the forces on the left of A, and z the perpendicular distance of P_h from A, we have

$$M = M_p - P_h z.$$

Since the work of deformation due to the shear and axial load is small, it may be neglected in comparison with that due to the bending moment. Under this assumption the work of deformation is

$$W = \frac{1}{2}\int \frac{M^2}{EI}\,ds = \frac{1}{2}\int \frac{(M_p - P_h z)^2}{EI}\,ds,$$

in which the integral is to be extended over the entire length of the rib. Applying the principle of least work to this expression, the partial derivative of W with respect to the unknown quantity P_h must be zero. Hence

$$\frac{\partial W}{\partial P_h} = -\int \frac{(M_p - P_h z)}{EI} z ds = 0,$$

or

$$-\int \frac{M_p z}{EI} ds + P_h \int \frac{z^2}{EI} ds = 0;$$

whence

$$P_h = \frac{\int \frac{M_p z}{EI} ds}{\int \frac{z^2}{EI} ds}.$$

If E is constant throughout the rib, this reduces to

$$P_h = \frac{\int \frac{M_p z}{I} ds}{\int \frac{z^2}{I} ds}.$$

The pole distance P_h found from this formula is the third condition necessary for the complete determination of the equilibrium polygon.

160. Second method of calculating the pole distance. The value of the pole distance P_h of the force diagram can also be calculated by assuming that the bending of the rib produces no change in the span. To apply this condition, the change in length which the span would naturally undergo is calculated and equated to zero.

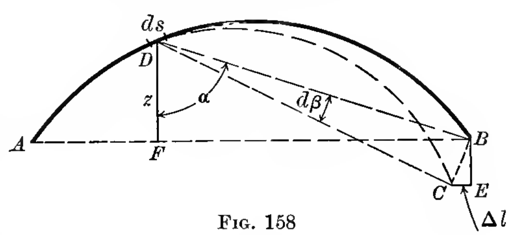


FIG. 158

Consider a small portion ds of the rib. If, for the moment, the rest of the rib is regarded as rigid, the bending of this portion would make the end B revolve about D as a center to a position at C (Fig. 158). Let $d\beta$ denote the angle between DB and DC, α the angle between DB and a vertical through D, z the ordinate DF, and CE, or Δl, the change in length of the span. Then

$$BC = DB \cdot d\beta, \quad DB \cos \alpha = z, \quad \text{and} \quad \Delta l = CB \cos \alpha.$$

Hence

$$\Delta l = DB \quad d\beta \cos \alpha = z d\beta.$$

From Article 66, the angular deformation $d\beta$ is given by the expression

$$d\beta = \frac{Mds}{EI}.$$

Consequently

$$\Delta l = zd\beta = \frac{Mz}{EI}ds,$$

and hence the total change in length of the span is*

$$l = \int \frac{Mz}{EI} \cdot ds.$$

Therefore the condition that the span shall be unchanged in length by the strain is

$$\int \frac{Mz}{EI} ds = 0.$$

The bending moment M in this expression has the same value as in the preceding article, namely, $M = M_p - P_h z$. Inserting this value of M in the above condition, it becomes

$$\int \frac{(M_p - P_h z)}{EI} zds = 0,$$

from which, as in the preceding article,

$$P_h = \frac{\int \frac{M_p z}{I} ds}{\int \frac{z^2}{I} ds}.$$

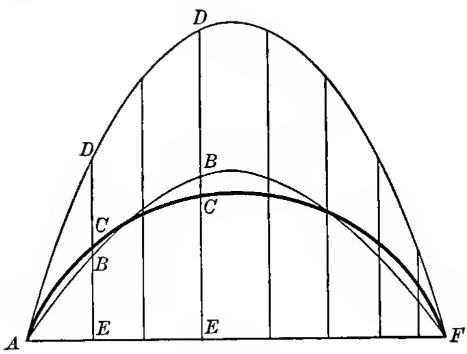


FIG. 159

161. Graphical determination of the linear arch. From the condition that the bending stress shall produce no change in the length of the span, the position of the linear arch may be determined graphically as follows.

In Fig. 159 let ACF represent the center line of the rib, ADF the corresponding equilibrium polygon drawn to any convenient scale, and ABF the linear arch. Then the linear arch can be obtained from the equilibrium polygon by reducing each ordinate of the latter in a certain

* The effects of changes of temperature and also of direct compressive stress in altering the length of the span are neglected, as they are slight in comparison with that due to bending strain.

ratio, say r. The problem then is to find this ratio r in which the ordinates to the equilibrium polygon must be reduced to give the linear arch.

The condition that the span is unchanged in length, derived in the preceding article, is

$$\int \frac{Mz}{EI} ds = 0,$$

in which z represents the ordinate CE to the rib, and ds an element of the rib. Since the bending moment M is proportional to the vertical intercept between the linear arch and the center line of the rib, this condition may be written

$$\int \frac{BC \cdot z}{EI} ds = 0;$$

or, since E may be assumed to be constant and

$$BC = BE - CE = BE - z,$$

this condition becomes

$$\int \frac{(BE - z)z}{I} ds = 0,$$

which may be written

$$\int \frac{BE \cdot z}{I} ds - \int \frac{z^2}{I} ds = 0.$$

If r denotes the ratio in which the ordinates to the equilibrium polygon must be decreased in order to give the linear arch, then

$$r = \frac{BE}{DE},$$

and consequently the condition becomes

$$\int \frac{r \cdot DE \cdot z}{I} ds - \int \frac{z^2}{I} ds = 0;$$

whence

$$r = \frac{\int \frac{z^2}{I} ds}{\int \frac{DE \cdot z}{I} ds}.$$

This expression for r can be evaluated graphically by replacing the integrals by summations and calculating the given functions for a

series of vertical sections taken at equal intervals *along the rib.* Thus, since ds in this case is constant,

$$r = \frac{\sum \frac{z^2}{I}}{\sum \frac{DE \cdot z}{I}},$$

in which the functions under the summation signs are to be calculated for each section separately, and their sum taken. After r has been found in this way the linear arch is obtained by decreasing the ordinates of the equilibrium polygon in the ratio $r : 1$, and the stress can then be calculated as explained in Article 157.

This method of determining the linear arch is due to Ewing.

162. Temperature stresses in two-hinged arched rib. When the temperature of an arched rib changes, the length of the rib also changes, and consequently stresses called **temperature stresses** are produced in the rib (compare Article 19). To calculate the amount of this stress let L denote the coefficient of linear expansion and T the change in temperature in degrees. Then each element of the rib of length ds changes its length by the amount $LTds$, the horizontal projection of which is $LTdx$. Therefore the total change l in the length of the rib is

$$l = \int_0^{2c} LTdx = 2\,cLT,$$

where $2\,c$ is the span. From Article 160, the total change in length of the span is given by the expression

$$l = \int_0^l \frac{Mz}{EI} ds.$$

Therefore

$$\int_0^l \frac{Mz}{EI} ds = 2\,cLT.$$

To simplify this expression assume that the modulus of elasticity E is constant throughout the rib, and that the moment of inertia I increases towards the abutments in the ratio $\frac{ds}{dx}$. Under this assumption $I = I_0 \frac{ds}{dx}$, where I_0 denotes the moment of inertia at the crown, and the above equation becomes

$$\frac{1}{EI_0} \int_0^{2c} Mzdx = 2\,cLT.$$

The only forces which tend to resist the change in length of the rib due to temperature stresses are the horizontal reactions P_h of the abutments. Therefore the external moment at any section of the rib with ordinate z is $M = P_h z$, and substituting this value in the above integral, it becomes

$$\frac{P_h}{EI_0}\int_0^{2c} z^2 dx = 2\,cLT;$$

whence

$$P_h = \frac{2\,EI_0 cLT}{\int_0^{2c} z^2 dx}.$$

This expression is easily evaluated in any given case, thus determining P_h and consequently the linear arch. The temperature stresses can then be calculated by the methods explained above, and combined

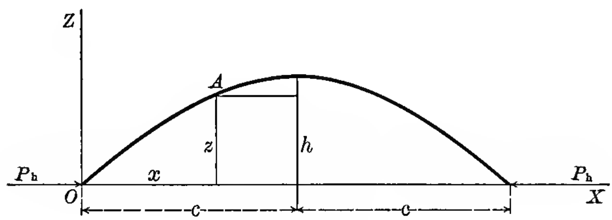


FIG. 160

with those due to the given loading. For a rise in temperature above that for which the arch was designed, T is positive and the horizontal reactions P_h of the abutments act inwardly; for a fall in temperature T is negative and the reactions P_h act outwardly.

To illustrate what precedes, the above formula will now be applied to a parabolic arched rib, which on account of its simplicity is the form ordinarily assumed in designing. Let h denote the rise of the arch, $2\,c$ its span, and x, z the coördinates of any point A on the rib (Fig. 160). Then, from the intrinsic property of the parabola, it follows that

$$\frac{h - z}{h} = \frac{(c - x)^2}{c^2};$$

whence

$$z = \frac{hx}{c^2}(2\,c - x).$$

Substituting this value of z in the integral $\int_0^{2c} z^2 dx$ and integrating, we have

$$\int_0^{2c} z^2 dx = \int_0^{2c} \left[\frac{hx}{c^2}(2c - x)\right]^2 dx = \frac{16\,ch^2}{15}.$$

Consequently for a parabolic arched rib the horizontal reaction of the abutments due to a change in temperature of T degrees is

$$P_h = \frac{2\,EI_0 cLT}{\frac{16\,ch^2}{15}} = \frac{15\,EI_0 LT}{8\,h^2}.$$

163. Continuous arched rib fixed at both ends. For a continuous arched rib fixed at both ends the problem of constructing the equilibrium polygon is subject to a threefold indetermination, since none of the three conditions necessary for its determination are given. The theoretical solution of the question by the principle of least work is as follows.

Let the vertical reaction R_1, the horizontal reaction P_h, and the bending moment M_0 at the left support be chosen as the three unknown quantities necessary to determine the linear arch. For a system of concentrated loads the moment M at any section of the rib distant x from the left support is

$$M = M_0 + Rx - P_h z - \sum_0^x P(x - d),$$

in which d is the distance of any load P from the left support, and the summation is to be extended over all the loads between the left support and the point under consideration. Similarly, for a uniform load of amount w per unit of length,

$$M = M_0 + Rx - P_h z - \frac{wx^2}{2}.$$

Now, from Article 73, the work of deformation W is given by the expression

$$W = \frac{1}{2}\int \frac{M^2}{EI} ds,$$

in which M has the value given by one or the other of the above expressions, depending on whether the loading is concentrated or uniform. To apply Castigliano's theorem to this expression it is

necessary to find the partial derivatives of W with respect to M_0, R_1, and P_h respectively, and equate these derivatives to zero. The three conditions obtained in this way are

$$(103)\qquad \begin{aligned} \frac{\partial W}{\partial M_0} &= \int \frac{M}{EI} \cdot \frac{\partial M}{\partial M_0} ds = \int \frac{M}{EI} ds = 0, \\ \frac{\partial W}{\partial R} &= \int \frac{M}{EI} \cdot \frac{\partial M}{\partial R} ds = \int \frac{Mx}{EI} ds = 0, \\ \frac{\partial W}{\partial P_h} &= \int \frac{M}{EI} \cdot \frac{\partial M}{\partial P_h} ds = -\int \frac{Mz}{EI} ds = 0, \end{aligned}$$

since from either of the above expressions for M we have $\frac{\partial M}{\partial M_0} = 1$, $\frac{\partial M}{\partial R} = x$, and $\frac{\partial M}{\partial P_h} = -z$. Inserting in these three conditions the value of M for the given form of loading, three simultaneous equations are obtained which may be solved for the three unknown quantities R_1, P_h, and M_0.

Equations (**103**) can also be obtained by assuming as our three conditions that the horizontal and vertical deflections of the supports are zero, and that the direction of the rib at the ends remains unchanged. The method of obtaining equations (**103**) from these assumptions is simply an extension of that given in Article 160 for the two-hinged arched rib.

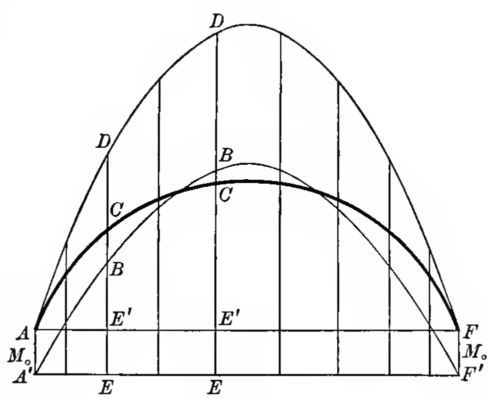


FIG. 161

164. Graphical determination of the linear arch for continuous arched rib. The simplest method of applying equations (**103**) to the determination of the linear arch is by means of a graphical treatment similar to that given in Article 161.

Consider first the case of symmetrical loading. Then if M_0 denotes the bending moment at either abutment, the linear arch has the same

form as for a rib with two hinges, except that its base is shoved down a distance M_0 below the springing line of the rib. Therefore in this case the linear arch is completely determined by the two quantities M_0 and r, the third condition being supplied by the symmetry of the figure.

In Fig. 161 let ACF represent the center line of the rib, $A'BF'$ the linear arch, and ADF the equilibrium polygon for the given system of loads. Since the bending moment M at any point of the rib is the vertical intercept BC between the linear arch and the center line of the rib, we have

$$M = BC = BE - CE' - EE',$$

or, since $BE = r \cdot DE'$,
$$M = r \cdot DE' - z - M_0.$$

Substituting this value of M in the first and third of equations (**103**), they become

$$\int \frac{rDE'}{EI} ds - \int \frac{z}{EI} ds - \int \frac{M_0}{EI} ds = 0,$$

and

$$\int \frac{rDE'zds}{EI} - \int \frac{z^2}{EI} ds - \int \frac{M_0 z}{EI} ds = 0.$$

If the expressions under these integral signs are evaluated for a number of vertical sections taken at equal distances *along the rib,* and the results are summed, we obtain the two conditions

$$r\sum \frac{DE'}{I} - \sum \frac{z}{I} - M_0 \sum \frac{1}{I} = 0,$$

and

$$r\sum \frac{DE' \cdot z}{I} - \sum \frac{z^2}{I} - M_0 \sum \frac{z}{I} = 0,$$

from which r and M_0 can easily be determined. The linear arch is then constructed by starting from a point at a distance M_0 below the left support, and decreasing the ordinates to the equilibrium polygon in the ratio $r : 1$.

If the loading is unsymmetrical, the moments at the ends of the rib are not equal. Let M_1 and M_2 denote the moments at the left and right ends respectively (Fig. 162). As before, the moment M at any point of the rib is the vertical intercept BC between the linear arch $A'BF'$ and the center line of the rib ACF. Consequently

$$M = BC = BE - CE' - EE'.$$

In this case, however, the distance EE' is not constant from A to F, but varies as the ordinates to a triangle, being equal to M_1 at A and to M_2 at F. Hence, for a point at a distance x from A,

$$(EE')_x = M_1 - \frac{x}{2c}(M_1 - M_2),$$

where $2c$ is the length of the span. Also $BE = r \cdot DE$, and $CE' = z$. Therefore

$$M = r \cdot DE - z - M_1 + \frac{x}{2c}(M_1 - M_2).$$

Let this value of M be inserted in equations (**103**). Then, if the expressions under the integral signs are evaluated for a number of

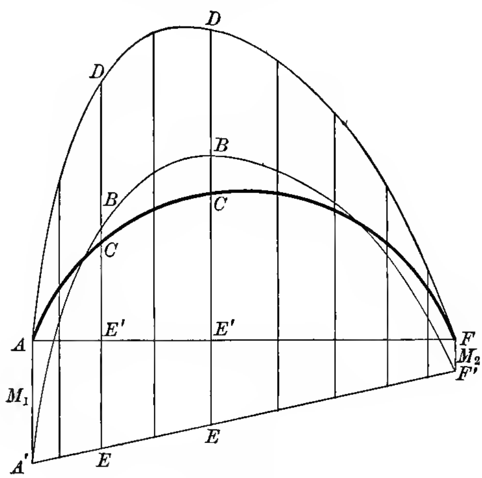


FIG. 162

vertical sections taken at equal distances *along the center line of the rib*, and their sums taken, the integrations in equations (**103**) can be replaced by summations giving the three conditions

$$r\sum\frac{DE}{I} - \sum\frac{z}{I} - M_1\sum\frac{1}{I} + \frac{M_1 - M_2}{2c}\sum\frac{x}{I} = 0,$$

$$r\sum\frac{DE \cdot x}{I} - \sum\frac{zx}{I} - M_1\sum\frac{x}{I} + \frac{M_1 - M_2}{2c}\sum\frac{x^2}{I} = 0,$$

$$r\sum\frac{DE \cdot z}{I} - \sum\frac{z^2}{I} - M_1\sum\frac{z}{I} + \frac{M_1 - M_2}{2c}\sum\frac{zx}{I} = 0.$$

Solving these three equations simultaneously for M_1, M_2, and r, the linear arch is constructed by laying off M_1 and M_2 from A and F respectively, and then reducing the ordinates to the equilibrium polygon in the ratio $r:1$, and laying them off from the line $A'F'$.

The stresses in the rib can then be calculated by the methods previously given (Article 157).

165. Temperature stresses in continuous arched rib. Using the notation of Article 162, the change in the length of the span due to a change in temperature of T degrees is

$$l = 2\,cLT.$$

Therefore, for temperature stresses equations (**103**) become

$$\int \frac{M}{EI}\,ds = 0, \quad \int \frac{Mx}{EI}\,ds = 0, \quad \int \frac{Mz}{EI}\,ds = -\,2\,cLT.$$

By hypothesis, the only external forces acting on the rib are the reactions and moments at the abutments due to the temperature stresses. Consequently, if R denotes the vertical reaction, P_h the horizontal reaction, and M_1 the moment at the left abutment, the moment M at any other point of the rib is

$$M = M_1 + Rx - P_h z.$$

If, then, this value of M is inserted in the above integrals and the resulting equations solved simultaneously for M_1, R, and P_h, the linear arch is thereby determined.

CHAPTER XI

FOUNDATIONS AND RETAINING WALLS*

166. Bearing power of soils. Since the character of a foundation is dependent upon the nature of the soil on which it is to rest, it is necessary in designing a foundation to know with a reasonable degree of accuracy the maximum load which the soil can sustain per unit of area without appreciable settlement; or, in other words, what is known as the **bearing power** of the soil.†

Ordinarily the results of previous experience are relied upon to give an approximate value of the bearing power of any given soil, and stability is assured by the adoption of a large factor of safety. For structures of unusual importance, however, or when the nature of the soil is uncertain, the results of previous experience are usually insufficient to assure stability, and special tests are necessary for the determination of the bearing power of the soil in question. Among notable structures for which such special tests have been made may be mentioned the State Capitol at Albany, N.Y.; the Congressional Library at Washington, D.C.; the suspension bridges at Brooklyn, N.Y., and at Cincinnati, Ohio; the Washington Monument; the Tower Bridge, London, etc.

By averaging the results of a large number of such tests, reliable information is furnished as to the bearing power of soils in general. The most commonly accepted of such average values are those given by Professor I. O. Baker in his *Treatise on Masonry Construction*, and are as shown in the table on the following page. Other values in common use are also quoted for comparison, and may be accepted as representative of modern practice.

* For a more detailed treatment of foundations and retaining walls the following special treatises may be consulted. Baker, *Treatise on Masonry Construction;* Howe, *Retaining Walls for Earth;* Fowler, *Ordinary Foundations;* Merriman, *Walls and Dams;* Patton, *Ordinary Foundations.*

† The bearing power of soils is analogous to what is called the crushing strength in the case of more rigid materials, such as stone and brick.

Material	Bearing Power tons/ft.²
Rock equal to best ashlar masonry	25
Rock equal to best brick masonry	15
Rock equal to poor brick masonry	5
Dry clay	4
Moderately dry clay	2
Soft clay	1
Cemented gravel and coarse sand	8
Compact and well-cemented sand	4
Clean, dry sand	2
Quicksand and alluvial soils	$\frac{1}{2}$

As an approximate working rule Trautwine recommends from 2 to 3 tons/ft.² as a safe load for compact gravel, sand, or loam, and from 4 to 6 tons/ft.² if a few inches of settlement may be allowed.*

The building laws of Greater New York may also be regarded as competent authority, and specify the following values.

Material	Bearing Power tons/ft.²
Firm, coarse sand, stiff gravel, or hard clay	4
Loam, clay, or fine sand, firm and dry	3
Ordinary clay and sand together, wet and springy	2
Soft clay	1

As a supplement to the above, these laws also specify that when foundations are carried down through earth by piers of stone or brick, or by concrete in caissons, the loads on same shall not exceed 15 tons/ft.² when carried down to rock, or 10 tons/ft.² when carried down to firm gravel or hard clay.

In order to obviate too large or expensive a foundation, it is often desirable to increase the bearing power of the soil. This may be accomplished in various ways.

Since, in general, soils are more condensed at greater depths, increasing the depth usually increases the bearing power of the soil.

* *Engineer's Pocket-Book*, 1902, p. 583.

In the case of wet or moist soils the same effect is obtained by drainage, as indicated in the tables on the preceding page.

A more marked increase in the bearing power may be obtained by excavating the soil and replacing it by a layer of moist sand; or by driving short piles and then either removing them and filling the hole immediately with moist sand, or else leaving the piles in the earth and covering them with a platform of timber or concrete.

When none of these methods will suffice, the soil must be excavated until a subsoil with an adequate bearing power is reached.

167. Angle of repose and coefficient of friction. When a mass of granular material, such as sand, gravel, or loose earth, is poured upon a level surface, the sides of the pile will assume a definite slope, called the *natural slope*. This maximum angle which the sides of the pile can be made to assume with the horizontal is called the **angle of repose**, and is a constant for any given material. Since the size of this angle is dependent upon the amount of friction between the particles of the material, it may be taken as a measure of the friction, or vice versa.

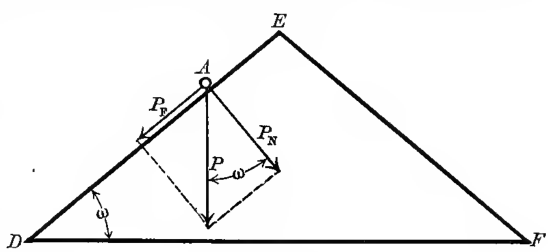


FIG. 163

The laws of friction as determined by experiment are that the force of friction is independent of the areas in contact, is dependent on the nature of the material, and is directly proportional to the normal pressure between the surfaces in contact. Let P_F denote the force of friction and P_N the normal pressure. Then the above laws may be expressed by the formula

$$P_F = kP_N,$$

where k is the constant of proportionality, and is called the **coefficient of friction.**

In Fig. 163 let DE represent the natural slope and ω the angle of repose, and consider a particle of the material of weight P at any point A in the natural slope. Let P be resolved into two components

P_F and P_N, respectively parallel and perpendicular to DE. Then $P_F = P_N \tan \omega$, and comparing this with the relation $P_F = kP_N$,

$$k = \tan \omega ;$$

that is to say, the coefficient of friction is equal to the tangent of the angle of repose.

The following table gives the numerical values of the angles of repose and coefficients of friction for various materials, and also the weight in pounds of one cubic foot of each material.*

Material	Angle of Repose ω	Coefficient of Friction $k = \tan \omega$	Weight lb./ft.³
Sand, dry and fine	28°	.532	110
" dry and coarse	30°	.577	95
" moist	40°	.839	110
" wet	30°	.577	125
Clay, damp	45°	1.000	125
" wet	15°	.268	150
Clayey gravel	45°	1.000	120
Shingle	42°	.900	. . .
Gravel	38°	.781	110
Alluvial soil	35°	.700	90
Peat	20°	.364	52
Concrete, best	. .	. . . . . .	160
" porous	.	. . . .	130
Brickwork	33°	.649	. .
pressed	. . .	. . . .	140
medium	. .	. . . .	125
soft	. . . .	. .	100
Masonry	31°	.601	. .
granite or limestone	. . . .	.	165
sandstone	. . . .	. . . . . .	144
mortar rubble	. . .	. .	154
dry rubble	. . . .	. . .	138

168. Bearing power of piles. The custom of driving piles into the soil to increase its bearing power is of very ancient origin, and is still frequently used because of its cheapness and efficiency. Until quite

* See Fanning, *Treatise on Hydraulic and Water Supply Engineering*, 15th ed., 1902, p. 345; Trautwine, *Engineer's Pocket-Book*, 1902, pp. 407–411; *Smithsonian Physical Tables*, 1896, Table 95; also the results compiled by Rankine from experiments by General Morin and others, *ibid.*, Table 149.

recently wood was the only material used for piles, and they were either driven by hand with sledges, or by means of a block, usually of metal, which was raised between two upright guides and allowed to fall on the head of the pile. The latter form of pile driver is still in frequent use for driving wooden piles, and is called the drop-hammer pile driver.

In 1839 Nasmyth invented the steam pile driver, which consists essentially of a steam cylinder supported vertically above the head of the pile by two uprights fastened to a cap which rests on the pile. The hammer in this case is a weight attached to the piston rod, and delivers a blow on the head of the pile at each stroke of the piston. The uprights which support the cylinder also serve as guides for the hammer, which varies in weight from 550 lb. to 4800 lb. This form of pile driver owes its efficiency to the rapidity with which the blows can be given, the number being from sixty to eighty per minute, thus preventing the soil from recovering its equilibrium between strokes, and greatly decreasing its resistance to penetration.

In modern engineering practice cast-iron and concrete piles are rapidly coming into use, and as neither of these materials is capable of standing repeated blows, piles of this kind are usually driven by means of an hydraulic jet. The jet is attached to the point of the pile, thus constantly excavating the soil in front of the pile as it descends, and enabling it to sink into place with little or no assistance other than its own weight.

The rational formulas in ordinary use for determining the bearing power of piles are based upon the assumption that the pile is driven by a drop-hammer pile driver, and express its bearing power in terms of the amount of penetration at the last blow. Since the bearing power of a pile is due in part to the friction of the earth on the sides of the pile, as well as to the resistance of the subsoil to penetration, and also since part of the energy of the hammer is absorbed by the friction of the guides, in compressing the head of the pile, in compressing the hammer, in overcoming the inertia of the pile, etc., a rigorous formula is too complicated to be of much practical value, although there are a number of elaborate discussions of the bearing power of piles which take all of these elements into consideration, notably the

theories of Rankine and Weisbach.* However, as several of the elements entering into the discussion are attended with considerable uncertainty, it is customary in practice to use either an empirical formula or the simple approximate formula deduced below, adopting a factor of safety large enough to cover the assumptions made.

Let P denote the weight of the hammer in pounds, h the height of the fall in inches, R the average resistance of the soil to penetration during the last blow in pounds, and d the penetration of the pile, due to the last blow, in inches. Then, assuming that all the work done by the hammer is expended in overcoming the resistance of the earth at the point of the pile, we have

$$Ph = Rd.$$

With a factor of safety of 6, the approximate formula for safe load on the pile becomes

$$R = \frac{Ph}{6d}. \tag{104}$$

As the head of a timber pile becomes "broomed" by repeated blows, and this greatly decreases the efficiency of the blow by absorbing the kinetic energy of the hammer, the head should be sawed off to a solid surface before making a test blow for determining the bearing power of the pile.

For a drop-hammer pile driver the empirical formula in most common use is

$$R = \frac{2Ph}{d+1}, \tag{105}$$

the notation being the same as above, and the factor of safety being 6. For a steam pile driver this formula becomes

$$R = \frac{2Ph}{d+0.1}, \tag{106}$$

where Ph represents the kinetic energy of the hammer.

The above empirical formula, (**105**) or (**106**), is commonly known as **Wellington's formula**, or the **Engineering News formula**, and has been incorporated in the building laws of Greater New York.

The only means of determining the bearing power of a pile driven by an hydraulic jet, is to observe the maximum load it can support without appreciable settlement.

* See Baker, *Treatise on Masonry Construction*, chap. xi.

Problem 289. A one-ton hammer falls 15 ft. on the head of a pile, and the settlement is observed to be .1 in. Calculate the safe load for the pile by formulas (**104**) and (**105**) and compare the results.

Problem 290. Under what conditions will the approximate rational formula (**104**) and the Engineering News formula (**105**) give substantially the same results?

Solution. If the values of R obtained from these two formulas were equal, then $\frac{Ph}{6d} = \frac{2Ph}{d+1}$; whence $d = \frac{1}{11}$ in. For other values of d the rational formula gives the greater value of the bearing power when $d < \frac{1}{11}$ in., and the empirical formula gives the greater value when $d > \frac{1}{11}$ in. From this it follows that the empirical formula is only applicable when the settlement at the last blow is small.

169. Ordinary foundations. Although the foundation of a structure is necessarily the first part to be constructed, it is the last part to be designed, for the weight of the structure determines the nature of the foundation, and this cannot be calculated until the structure has assumed definite proportions.

The load which a structure is designed to carry consists primarily of three parts.

1. The **dead load**, due to the weight of the structure and the permanent fixtures, such as plumbing and heating apparatus, elevators, water tanks, machinery, etc.

2. The **live load**, which depends on the use to which the structure is to be put, and which may vary from 20 lb./ft.^{2} to 400 lb./ft.^{2}

3. The **wind load**, due to the overturning action of the wind upon the side of the structure. These three parts of the total load must be calculated separately and then combined so as to give the maximum resultant. The area of the foundation is then found at once by dividing this maximum load by the safe bearing power of the soil.

The chief concern in designing a foundation, however, is not that its settlement shall be zero, but that it shall be uniform throughout. For if one part of a foundation settles more than another, it is evident that cracks are bound to occur which will seriously weaken the structure and may even destroy its usefulness altogether. Since uniformity of settlement implies uniformity of pressure on the soil, the condition which determines the stability of a foundation and its superstructure is simply *uniformity of pressure on the soil.*

The effect of violating this condition is frequently seen, the most common instance being that of ordinary dwelling houses in which several openings, say a door and a number of windows, occur one

above another. It is evident in this case that if the foundation is of the same width throughout, the centers of pressure will fall outside the centers of resistance, which will tend to throw the top of the wall outward on either side, and so result in cracks between the openings (Fig. 164). The remedy for this is either to narrow the foundation, or omit it altogether under the openings, or else extend it beyond the ends of the wall, the length of this extension being of such amount that the centers of pressure will fall *inside*, or at least coincide with, the centers of resistance.

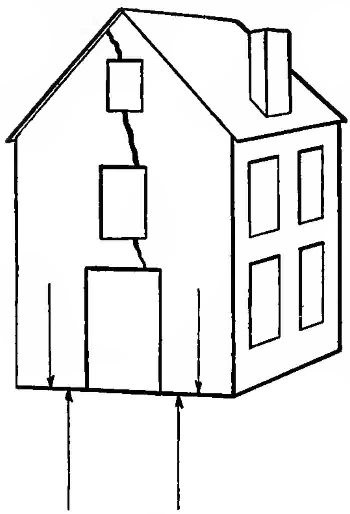


FIG. 164

When a foundation extends beyond the ends of a wall the projection is called the **footing**. To dimension the footing it may be regarded as a simple cantilever, and its thickness calculated by the ordinary theory of beams. Thus let h denote the thickness of the footing in inches for a concrete foundation, or the thickness of the bottom footing course in inches for a masonry foundation, b the width of the footing in inches, u the ultimate strength of the material in lb./in.^{2}, and P the load in tons/ft.^{2} Then, since 1 ton/ft.^{2} = 13.9 lb./in.^{2}, the moment at the face of the wall is

$$M = (13.9\,Pbx) \cdot \frac{x}{2};$$

or, since $I = \dfrac{bh^3}{12}$ and $u = \dfrac{M \cdot \frac{h}{2}}{I}$, we have $u = \dfrac{41.7\,x^2 P}{h^2}$; whence

$$\boldsymbol{h = 6.45\,x\sqrt{\frac{P}{u}}}, \text{ approximately.}$$

Problem 291. Find the thickness of the bottom footing course for a masonry foundation if the load is 1 ton/ft.^{2}, the factor of safety is 10, the footing is to extend 18 in. beyond the face of the wall, and is composed of limestone for which u = 15,000 lb./in.^{2}

170. Column footings. In the modern construction of tall buildings the design frequently provides that the entire weight of the building and its contents shall be carried by a steel framework of columns and girders. This "skeleton type" of tall-building construction, as it is called, necessitates a new type of foundation, since each column load must be calculated separately and transmitted to the soil by a footing of sufficient size to give the necessary amount of bearing area.

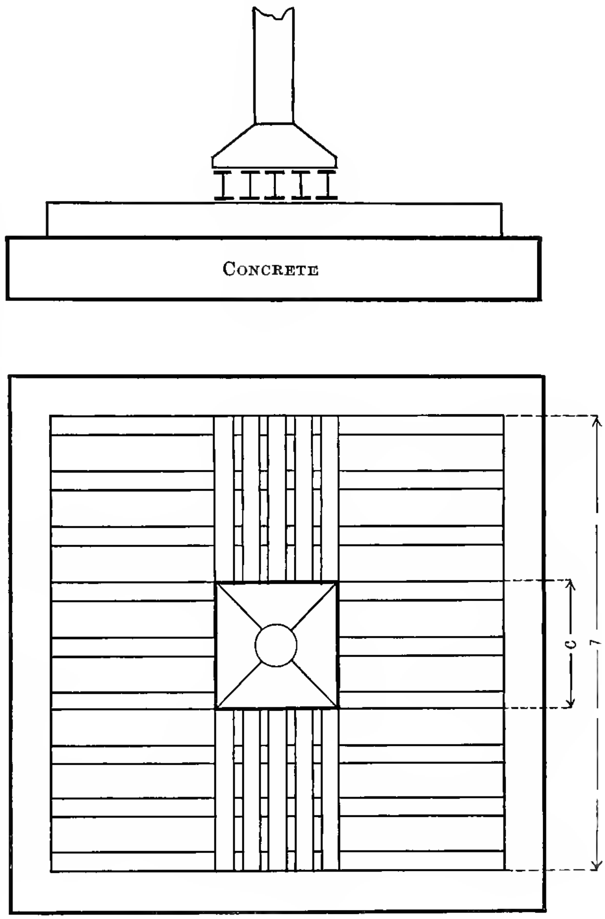


FIG. 165

If the columns reach solid rock, the footing may consist simply of a base plate of such form as to give the column a solid bearing and afford sufficient anchorage to prevent the footing from lateral movement.

For compressible soils the column is usually supported by a cast-iron base plate resting on a footing consisting of two or more layers of steel rails or I-beams, the whole resting on a concrete base, as shown in Fig. 165.

What has been said in the preceding article in regard to the calculation of the loads carried by the foundation also applies to the calculation of column loads, and the method of designing a column footing is essentially the same as for a masonry footing, explained above. Thus let P denote the total column load in tons, c the length of one side of the base plate in inches, and l the length in inches of the beams supporting it (Fig. 165). Then, if the base plate is assumed to be stiff enough to carry the load on its perimeter, the maximum moment M will occur at one edge of the base plate. Since the reaction on one side of the base plate is $2000\,P \cdot \frac{l-c}{2\,l}$, the amount of this moment is

$$M = \frac{2000\,P(l-c)}{2\,l} \cdot \frac{l-c}{4} = \frac{250\,P(l-c)^2}{l} \text{ in. lb.}$$

Consequently, if n is the number of beams supporting the base plate, the maximum moment for one beam is

$$M_1 = \frac{250\,P(l-c)^2}{nl} \text{ in. lb.}$$

If the base plate is assumed to be only stiff enough to distribute the load uniformly, the maximum moment will occur at the center of the beams, and its value will be (cf. Article 52 (E))

$$M = \frac{2000\,P\left(l - \frac{c}{2}\right)}{4} = 250\,P(2\,l - c) \text{ in. lb.}$$

In this case the maximum moment for one beam is

$$M_1 = \frac{250\,P(2\,l-c)}{n} \text{ in. lb.}$$

Now let p denote the allowable fiber stress per square inch, I the moment of inertia of a cross section of one beam, and e half the depth of the beam. Then the moment of resistance of one beam is

$$M = \frac{pI}{e}.$$

For foundation work p is usually taken to be 20,000 lb./in.^{2} Substituting this value, the moment of resistance becomes

$$M = 20{,}000\,\frac{I}{e} = 20{,}000\,S,$$

where S denotes the section modulus. Equating the moment of resistance to the external bending moment and solving the resulting equation for S, we have in the first case

$$S = \frac{P(l-c)^2}{80\,ln},$$

and in the second case

$$S = \frac{P(2\,l-c)}{80\,n}.$$

In designing a column footing the column load P is first calculated, and the area of the footing determined by dividing the column load by the safe bearing power of the soil. The size of base plate and number of beams supporting it are next assumed, and the section modulus calculated by one of the above formulas. The size of beam to be used is then determined by choosing from the tables a beam whose section modulus agrees most closely with the calculated value of S.

Problem 292. Design the footing for a column supporting a load of 400 tons, and resting on a base plate 4 ft. square, so that the pressure on the foundation bed shall not exceed 3 tons/ft.^{2}

171. Maximum earth pressure against retaining walls. A wall of concrete or masonry built to sustain a bank of earth, or other loose material, is called a **retaining wall.**

In Chapter X it was shown that in order to determine the stability of an arch three conditions were necessary, which might conveniently be chosen as the direction, amount, and point of application of the resultant pressure on any cross section of the arch ring. The same necessity arises in the discussion of retaining walls, namely, that three conditions are necessary for the complete solution of the problem, and a number of theories have been advanced, notably those of Coulomb, Weyrauch, and Rankine, based on different assumptions as to these conditions.

All theories, however, agree upon two of these assumptions, namely, (1) that the pressure against the wall is due to a wedge of earth, or, in other words, that the surface along which the earth tends to slide against the wall is a plane; and (2) that the point of application of the resultant earth pressure is one third of the height of the wall from the bottom. Neither of these assumptions is rigorously correct, for the first is equivalent to neglecting the cohesion of the earth, and

the second assumes that the earth pressure against the wall is the same as if the earth was a liquid. However, the uncertainty attending the exact degree of homogeneity of the materials under consideration probably does not warrant any greater precision in these first two assumptions.

The third assumption relates to the *direction* of the maximum pressure, and is the point on which the various theories differ. Thus Coulomb and Weyrauch assume that the pressure is normal to the

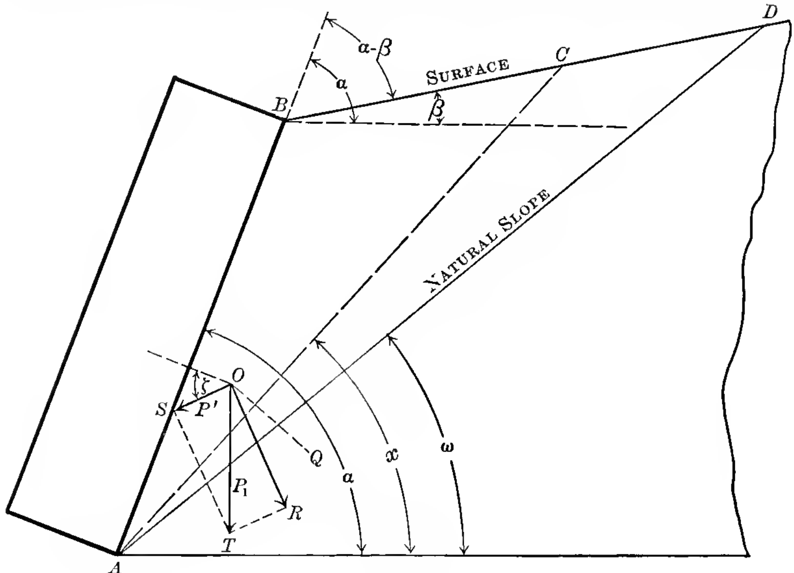


FIG. 166

back of the wall; Rankine assumes that it makes an angle with the back of the wall equal to the angle of repose of the material; while other authorities assume values intermediate between these two.

In the present discussion the first two conditions mentioned above will be retained, and the third condition will be replaced by the assumption that the resultant earth pressure makes an unknown angle ζ with a normal to the back of the wall. The assumptions are, then:

1. The surface of rupture is a plane.
2. The point of application of the resultant pressure is one third of the height of the wall from the bottom.

3. The resultant pressure is inclined at an angle ζ to a normal to the back of the wall.

From the result of the theory based on these assumptions, the values of the resultant earth pressure given by Coulomb, Weyrauch, Rankine, and others will then be deduced as special cases by giving different values to ζ.

In Fig. 166 let AB represent the back of the wall, BD the surface of the ground, AD the natural slope, and AC any line included between AB and AD. Also let P' denote the resultant pressure due to the wedge BAC, P_1 the weight of this wedge, OR its reaction against the plane AC, ζ the angle between P' and a normal to the back of the wall, ω the angle of repose of the earth, α the angle between the back of the wall and the horizontal, β the angle between the surface of the ground and the horizontal, and x the angle between AC and the horizontal.

Then in the triangle TOS, by the law of sines,

$$P' = P_1 \frac{\sin OTS}{\sin OST};$$

or, since $TOR = x - \omega$ and $TOS = 180° - \alpha - \zeta$, we have $OST = \alpha + \zeta - x + \omega$, and, consequently,

$$P' = P_1 \frac{\sin(x - \omega)}{\sin(\alpha + \zeta + \omega - x)}.$$

To find an expression for P_1, let w denote the weight of a unit volume of the material, say the weight of one cubic foot. Then for a section of unit length in the direction of the wall

$$P_1 = w(\text{area } ABC) = \frac{w}{2} AB \cdot AC \sin BAC;$$

or, if h denotes the height of the wall, $AB = \dfrac{h}{\sin \alpha}$, $BAC = \alpha - x$, and $AC = AB \dfrac{\sin(\alpha - \beta)}{\sin(x - \beta)}$; whence

$$P_1 = \frac{wh^2 \sin(\alpha - \beta) \sin(\alpha - x)}{2 \sin^2 \alpha \sin(x - \beta)},$$

and, consequently,

$$P' = \frac{wh^2 \sin(\alpha - \beta) \sin(\alpha - x) \sin(x - \omega)}{2 \sin^2 \alpha \sin(x - \beta) \sin(\alpha + \zeta + \omega - x)}.$$

The problem now consists in finding the value of the variable angle x for which P' is a maximum, which may be expressed symbolically by the conditions

$$\frac{dP'}{dx} = 0 \quad \text{and} \quad \frac{d^2P'}{dx^2} < 0.$$

In order to reduce the expression for P' to a form more suitable for differentiation, we make use of the following identity.

$$\begin{aligned}\cot(\alpha - x) - \cot(\alpha - \omega) &= \frac{\cos(\alpha - x)}{\sin(\alpha - x)} - \frac{\cos(\alpha - \omega)}{\sin(\alpha - \omega)} \\ &= \frac{\cos(\alpha - x)\sin(\alpha - \omega) - \cos(\alpha - \omega)\sin(\alpha - x)}{\sin(\alpha - x)\sin(\alpha - \omega)} \\ &= \frac{\sin(x - \omega)}{\sin(\alpha - x)\sin(\alpha - \omega)};\end{aligned}$$

whence

$$\sin(x - \omega) = \sin(\alpha - x)\sin(\alpha - \omega)[\cot(\alpha - x) - \cot(\alpha - \omega)].$$

Similarly,

$$\sin(x - \beta) = \sin(\alpha - x)\sin(\alpha - \beta)[\cot(\alpha - x) - \cot(\alpha - \beta)],$$

and

$$\sin(\alpha + \omega + \zeta - x) = \sin(\alpha - x)\sin(\omega + \zeta)[\cot(\alpha - x) - \cot(\omega + \zeta)].$$

Substituting these values in the expression for P', the latter becomes

$$P' = \frac{wh^2\sin(\alpha - \omega)}{2\sin^2\alpha\sin(\omega + \zeta)} \cdot \frac{\cot(\alpha - x) - \cot(\alpha - \omega)}{[\cot(\alpha - x) - \cot(\alpha - \beta)][\cot(\alpha - x) + \cot(\omega + \zeta)]}.$$

Now the terms in this expression which contain the variable x are all of the same form, namely, $\cot(\alpha - x)$. This term may therefore be replaced by a new variable y, and the remaining terms by letters denoting constants. Thus let

$$\cot(\alpha - x) = y, \quad \frac{wh^2\sin(\alpha - \omega)}{2\sin^2\alpha\sin(\omega + \zeta)} = A,$$

$$\cot(\alpha - \omega) = B, \quad \cot(\alpha - \beta) = C, \quad \cot(\omega + \zeta) = D.$$

Then

$$P' = A\frac{y - B}{(y - C)(y + D)}.$$

Equating to zero the first derivative of P' with respect to y, we have

$$\frac{dP'}{dy} = A\frac{(y - C)(y + D) - (y - B)(y - C) - (y - B)(y + D)}{(y - C)^2(y + D)^2} = 0;$$

whence the condition for a maximum is

$$y = B + \sqrt{(B - C)(B + D)}.$$

Substituting this value of y in the expression for P', the latter becomes

$$P'_{\max} = \frac{A}{(\sqrt{B-C}+\sqrt{B+D})^2} = \frac{A}{(B+D)\left(1+\sqrt{\dfrac{B-C}{B+D}}\right)^2};$$

or, replacing A, B, C, D by their values,

$$(107) \quad P'_{\max} = \frac{wh^2 \sin^2(\alpha-\omega)}{2\sin^2\alpha \sin(\alpha+\zeta)} \cdot \frac{1}{\left(1+\sqrt{\dfrac{\sin(\omega-\beta)\sin(\omega+\zeta)}{\sin(\alpha-\beta)\sin(\alpha+\zeta)}}\right)^2},$$

which is the general formula for the maximum inclined earth pressure against retaining walls.

The various standard theories as to the maximum earth pressure may now be obtained as special cases of the above general formula by making the following assumptions.*

1. **Weyrauch's formula.** Assume that the pressure is normal to the back of the wall. Then $\zeta = 0$, and formula (**107**) becomes

$$P'_{\max} = \frac{wh^2 \sin^2(\alpha-\omega)}{2\sin^3\alpha\left(1+\sqrt{\dfrac{\sin(\omega-\beta)\sin\omega}{\sin(\alpha-\beta)\sin\alpha}}\right)^2}.$$

2. **Rankine's formula.** Assume that the angle of repose of earth on masonry is equal to the angle of repose of earth on earth. Then $\zeta = \omega$, and formula (**107**) becomes

$$P'_{\max} = \frac{wh^2 \sin^2(\alpha-\omega)}{2\sin^2\alpha\sin(\alpha+\omega)} \cdot \frac{1}{\left(1+\sqrt{\dfrac{\sin(\omega-\beta)\sin 2\omega}{\sin(\alpha-\beta)\sin(\alpha+\omega)}}\right)^2}.$$

3. **Poncelet's formula.** In Rankine's formula assume that the earth surface is horizontal and the back of the wall is vertical. Then $\beta = 0°$ and $\alpha = 90°$, and the preceding formula becomes

$$P'_{\max} = \frac{wh^2 \cos\omega}{2(1+\sqrt{2}\sin\omega)^2}.$$

4. **Coulomb's formula.** Assume, as in 3, that the earth surface is horizontal and the back of the wall vertical, and make the further

* It is not intended to convey the idea that Weyrauch, Rankine, etc., made these assumptions explicitly, but that they lead to formulas identical with theirs.

assumption that the pressure is normal to the back of the wall. Then $\beta = 0$, $\alpha = 90°$, $\zeta = 0$, and formula (**107**) becomes

$$P'_{\max} = \frac{wh^2}{2}\tan^2\left(45° - \frac{1}{2}\omega\right).$$

5. **Rankine's formula for vertical wall.** Assume that the back of the wall is vertical and that the line of action of the resultant earth pressure is parallel to the surface of the earth. Then $\alpha = 90°$, $\zeta = 90° + \beta - \alpha$, and formula (**107**) becomes

$$P'_{\max} = \frac{wh^2\cos^2\omega}{2\cos\beta\left(1 + \sqrt{\dfrac{\sin(\omega+\beta)\sin(\omega-\beta)}{\cos^2\beta}}\right)^2}.$$

6. **Maximum normal pressure.** Assume that β has its maximum value, which will be when $\beta = \omega$. Then Weyrauch's formula becomes

$$P'_{\max} = \frac{wh^2\sin^2(\alpha-\omega)}{2\sin^3\alpha},$$

which is the greatest normal thrust that can be caused by a sloping bank.

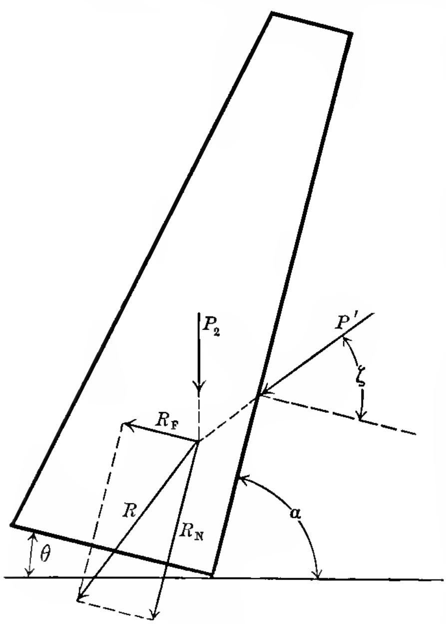


FIG. 167

Problem 293. A wall 20 ft. high is inclined at an angle of 85° to the horizontal and supports a backing of clayey gravel the surface of which makes an angle of 20° with the horizontal. Compute the maximum pressure against the back of the wall by Weyrauch's and Rankine's formulas, and compare the results.

Problem 294. By the use of Poncelet's formula compute the maximum pressure in the preceding problem if the back of the wall is vertical and the surface of the ground is horizontal.

Problem 295. What is the greatest normal pressure that can be caused by a bank of loose sand against a vertical wall 18 ft. high?

172. Stability of retaining walls. The conditions for the stability of a retaining wall are the same as those given in Article 155 for the

stability of abutments, namely, that the wall must be secure against sliding on its base and against overturning.

Let P_2 denote the weight of the wall, P' the resultant earth pressure, and R the resultant of P_2 and P' (Fig. 167). Then, if R is resolved into two components R_F and R_N, respectively parallel and perpendicular to the base of the wall, the condition for stability against sliding is that R_F shall be less than the friction on the base, or, symbolically,

$$R_F < kR_N.$$

Let g denote the factor of safety. Then this condition may be written

$$R_F = \frac{kR_N}{g}. \tag{108}$$

To find the values of R_F and R_N, let P' and P_2 be resolved into components parallel to R_F and R_N respectively. Then, in the notation of the preceding article,

$$R_F = P' \sin(\alpha + \theta + \zeta) - P_2 \sin\theta,$$

and

$$R_N = P_2 \cos\theta - P' \cos(\alpha + \theta + \zeta).$$

Substituting these values of R_F and R_N in equation (**108**) and solving the resulting expression for g,

$$g = \frac{k[P_2 \cos\theta - P' \cos(\alpha + \theta + \zeta)]}{P' \sin(\alpha + \theta + \zeta) - P_2 \sin\theta}. \tag{109}$$

If the base of the wall is horizontal, $\theta = 0$ and equation (**109**) becomes

$$g = \frac{k[P_2 - P' \cos(\alpha + \zeta)]}{P' \sin(\alpha + \zeta)}. \tag{110}$$

For security against sliding the factor of safety should not be less than 3; consequently, the criterion for stability against sliding may be stated as

$$g \geqq 3,$$

where the value of g is calculated from equation (**109**) or (**110**).

In applying this criterion it should be noted that the value of ζ must first be assumed (Article 171; $0 \leqq \zeta \leqq \omega$).

The following table gives average values of the angle of repose and coefficient of friction of masonry on various substances.*

* See references at the foot of p. 246.

Material	Angle of Repose	Coefficient of Friction
Masonry on dry clay	27°	.510
" moist clay	18°	.325
" wet clay	15°	.268
" dry earth	30°	.577
" clayey gravel	30°	.577
" sand or gravel	35°	.700
" dry wooden platform . .	31°	.601
" wet wooden platform . . .	37°	.754
" masonry, dry	31°	.601
" masonry, damp mortar . . .	36°	.726

In order for a wall to fail by overturning, it must either rotate about the outer edge of the base or, in the case of a masonry wall, open at one of the joints. The cause of failure in both cases is the same, namely, that the stress on the base or joint is partly tensile. Consequently, the criterion for stability against overturning is that the resultant R must strike within the middle third of the base or joint, as the case may be (cf. Articles 62, 148, 2, and 155).

This criterion can best be applied graphically. Thus having assumed a value for the angle ζ, the resultant earth pressure P' is calculated from the formula in Article 171, corresponding to this assumption of ζ, and combined with the weight of the wall into a single resultant R. If this resultant does not strike within the middle third of the base, or within the middle third of all the joints in the case of a masonry wall, the design must be altered until the criterion is satisfied.

173. Thickness of retaining walls. In designing a retaining wall economy of material is secured by making the base of such thickness that the resultant R, obtained by combining the weight of the wall P_2 with the maximum earth pressure P', shall fall at the outer edge of the middle third. However, theoretical formulas for determining the least thickness consistent with this condition are too complicated to be of practical value, and for this reason the design is usually based on an empirical formula.

In railroad practice Trautwine recommends that for vertical walls of rectangular cross section, supporting loose sand, gravel, or earth

level with the top, the thickness b of the base of the wall in terms of its total height h should be as follows:*

For wall of cut stone or large ranged rubble† in mortar,

$$b = .35\,h.$$

For wall of good common scabbled mortar rubble, or brick,

$$b = .40\,h.$$

For wall of well scabbled dry rubble,

$$b = .50\,h.$$

These empirical rules may be regarded as representative of the best American practice, and may be used to give a first approximation in making a tentative design.

By inclining the wall backward the angle between the earth thrust P' and the wall is decreased, and consequently the resultant R is made to approach more nearly the center of the base. This allows the thickness of the base to be decreased and thus lessens the amount of material in the wall, although it slightly increases its depth. However, there is a restriction upon the amount of inclination which is permissible, for the inclination also has the effect of increasing the tendency to slide on the base or joints. In practice these considerations are balanced by inclining the back of the wall at a small angle, say 5° or 10°, to the vertical (i.e. $\alpha = 80°$ or $85°$), and at the same time cutting the footing into steps perpendicular to the line of action of the resultant R, thus securing economy of material without sacrificing stability.

The thickness of the top of the wall is determined by the necessity of providing for the lateral pressure of the earth, due to the action of frost. Since the action of frost is greatest near the top of the wall where the material is most exposed, it is likely to push the top over if the wall is made only thick enough to resist the pressure due to the weight of the earth. This consideration, therefore, limits the least thickness of the wall at the top to about two feet for masonry, or somewhat less than this amount for concrete, since the latter has no joints and therefore offers a greater moment of resistance.

* *Engineer's Pocket-Book*, 1902, p. 603.

† Masonry composed of rough, undressed stones is called *rubble;* scabbled rubble has the roughest irregularities knocked off with a hammer.

From the above, it follows that for an economical design the cross section of a wall should be trapezoidal, the thickness of the base being determined by the consideration of stability against overturning, and the thickness of the top by the maximum action of frost.

The inclination of either face of a wall to the horizontal is usually expressed by giving the ratio of the horizontal projection of this face to its vertical projection. This ratio is called the **batter**, and is given in inches of horizontal projection per foot of height. For example, if a wall makes an angle of $80\frac{1}{2}°$ with the horizontal, it is said to be "battered 2 to 1," since the ratio of its horizontal projection to its vertical projection is equal to cot α, and in the present case

$$\cot \alpha = \cot 80\tfrac{1}{2}° = .1673 = \tfrac{2}{12}, \text{ approximately.}$$

Problem 296. Design a concrete retaining wall to support a bank of loose earth 25 ft. high, the back of the wall to be inclined backward at a batter of $1\frac{1}{2}$ to 1.

PART II

PHYSICAL PROPERTIES OF MATERIALS

Part II

PHYSICAL PROPERTIES OF MATERIALS

CHAPTER XII

IRON AND STEEL

174. Introductory. A study of the properties of materials used in engineering construction involves a study of the machines used for making the tests and the method of conducting these tests. From the time of Galileo, in 1600 A.D., tests have been made to determine the strength of materials, but only during the past fifty years has any very great advance been made. The rapid development of the past half century has been due to the notable increase in the construction of large buildings, bridges, etc.; for where engineers were formerly content to use material without being tested, the importance of modern constructions demands that the physical properties of the materials used shall be determined for each large contract.

The early testing machine consisted of little more than an ordinary scalebeam with the test piece attached to one end and the load applied at the other. These were used for making tension tests, and machines equally as simple were used for compression and flexure tests. Fig. 168 shows a type of these machines which was used by Kirkaldy about 1860. The specimen to be tested was held in the jaws g while the lever F was in the position of the dotted lines (Fig. 168). A load N was then applied to the end of the lever and gradually increased until the specimen was ruptured.

Testing machines have been much improved during the past twenty or thirty years in the United States by Riehlé Bros. and Olsen & Co.,

both of Philadelphia, Pennsylvania. The machines as now constructed for ordinary testing purposes consist of a platform scales with the usual means of measuring loads, and a screw press operated by an outside source for applying the loads. Fig. 169 is a machine of 100,000 lb. capacity, built by Olsen & Co., and may be taken as a type. The four upright pieces *A* with the base *B* upon which they rest form the platform of the scales. This platform rests upon knife-edges *C* attached to a system of levers *D* which terminate finally in a graduated lever *E* (the scalebeam) provided with a movable poise. Each lever is supported by knife-edges resting upon hardened steel plates. The screw press in this case is seen in the

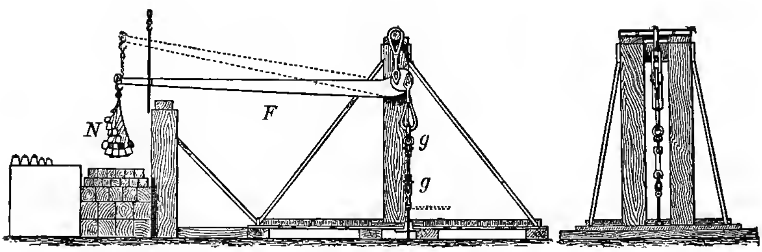


Fig. 168

four screws *F* with their movable crosshead *G*. The upper crosshead *H* is attached to the four upright pieces and is a part of the scale platform.

175. Tension tests. If a piece is to be tested in tension, one end is attached to the upper crosshead and the other end to the lower. The turning of the screws, due to the driving mechanism on the other side of the machine, causes the lower crosshead to move downward, thus bringing pressure to bear on the upper crosshead. From here it is transmitted to the base and thence to the levers, and is measured by movement of the poise on the graduated scalebeam. Machines of 20,000 lb., 30,000 lb., 50,000 lb., 100,000 lb., 200,000 lb., and 300,000 lb. capacity are manufactured, as well as a great many machines for making special tension tests. In the larger testing machines the upper head is usually adjustable so as to accommodate specimens of various lengths, but in the smaller machines the upper head is fixed.

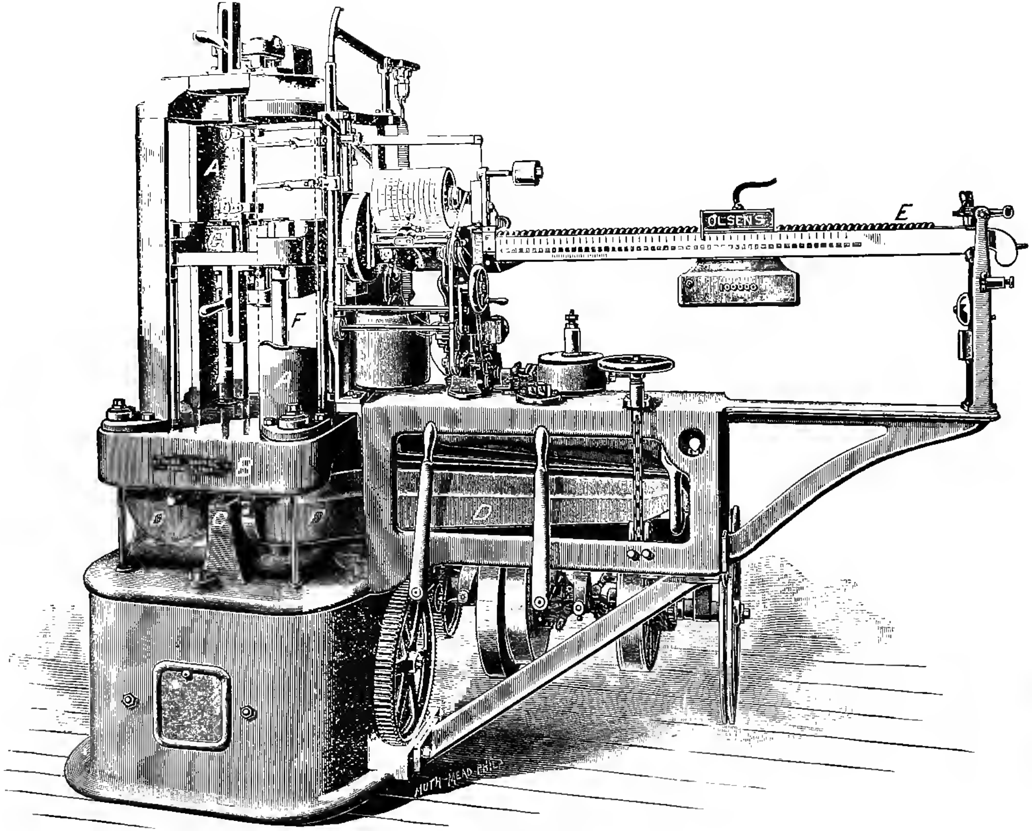


Fig. 169. — Improved Testing Machine

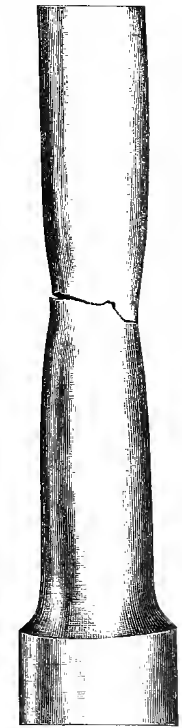


Fig. 170. — Tension Test Specimen

The **tensile strength** in pounds per square inch is computed by dividing the load read from the scalebeam by the area of cross section of the test specimen (see Article 20). Expressed as a formula,

$$\textbf{Tensile strength in lb./in.}^2 = \frac{\textbf{load from scalebeam}}{\textbf{area of cross section}}.$$

176. Compression tests. To make compression tests the piece is placed on a small block resting on the platform, and the lower crosshead, provided with a similar block, is brought down upon it. The further lowering of the crosshead compresses the specimen. The pressure comes on the platform through the crossbeam that rests upon it, and is transmitted to the scalebeam, where it is measured.

The **compressive strength** in lb./in.^{2} is computed by dividing the load in pounds as read on the scalebeam by the area of cross section of the test specimen, as in finding the tensile strength.

177. Flexure tests. Beams are tested in flexure by mounting the specimen on a crossbeam provided with knife-edges and applying the load from above by means of a knife-edge attached to the under side of the moving head. The beam is tested by lowering the moving head as in the compression tests.

The **fiber stress** in the outer fiber of the beam is computed in this case from the formula (see Article 52),

$$p = \frac{Ple}{4I},$$

where e is the distance from the neutral axis to the outer fiber, I is the moment of inertia with reference to the neutral axis, P is the load in pounds as read from the scalebeam, l is the length of the span in inches, and p is the fiber stress in lb./in.^{2}

The **maximum deflection** for the concentrated central load is computed by the formula (see Article 67),

$$D = \frac{Pl^3}{48\,EI},$$

where D is the deflection at the center, E the modulus of elasticity (see Article 8), and P, l, and I have the same meaning as above.

In case the beam is loaded at the third points, uniformly, eccentrically, or otherwise, the corresponding expressions are used for fiber stress and deflection (see Articles 52, 67).

178. Method of holding tension specimens. To make a tension test of a material a special test piece is usually provided. This test piece has the same composition as the rest of the material, but has a special form, being larger at the ends than in the central portion (see Article 20). Fig. 170 illustrates a test piece made from a carbon steel bar turned down in the central portion.* The machines are provided with serrated wedges for holding the large ends of the test piece, and as the load is applied these serrations sink into the specimen, thus holding it firmly.

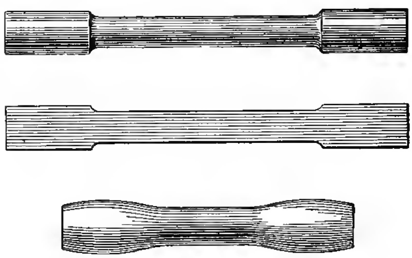


Fig. 171

The behavior of the specimen in tension is studied by noting the behavior of the reduced portion, which should be far enough from the ends so that the local stress caused by the wedges will have no effect upon it.

Flat pieces, such as pieces of boiler plate, are left as they come from the rolls on two sides, and the edges are machined to get the reduced cross section, as shown in Fig. 171. The lower specimen, of cast iron, is made with rounded corners to eliminate shrinkage stresses. Rolled material is often tested without being turned down. Special holders and clamps are usually provided for holding tension specimens of timber.

179. Behavior of iron and steel in tension. Wrought iron and mild steel when tested in tension conform to Hooke's law up to the elastic limit, a point which is usually well defined in these materials. They then suffer a rapid yielding, with little increase of load, reaching a point where the piece elongates very much for no increase of load. This point is known as the **yield point**. It is indicated by the scaling of the oxide from the specimen that has not been machined, and by the dropping of the beam of the testing machine, if it has been kept balanced up to this point. Beyond this point stress increases much

* Dimensions for standard test specimens of different materials are given in Article 203.

more slowly than deformation, until finally rupture is about to occur, at which point the load attains its maximum value, called the **ultimate load**. If the stress be continued, the piece begins to *neck* and breaks at a load somewhat less than the maximum (see Article 7). This necking is due to the fact that the metal under great strain becomes plastic and flows. Brittle materials, such as cast iron and hard steel, show very little, if any, necking. In computing the fiber stress at the maximum load the original cross section is used.

In commercial tests the load at the yield point (commercial elastic limit) and the maximum load are noted; also the percentage of elongation and the percentage of reduction of cross section. The percentage of elongation is the increase in length divided by the original length multiplied by 100. This percentage varies with the original length taken (see Article 20), and therefore is usually computed for an original length of eight inches. The percentage of reduction of cross section is the decrease in area of the cross section divided by the original area of the cross section multiplied by 100. In some commercial laboratories provision is made for making as many as sixty tests per hour on one machine.

180. Effect of overstrain on wrought iron and mild steel. If wrought iron and mild steel are strained just beyond the elastic limit in tension or compression, then released and tested again in the same direction, it has been found that this second test shows that the elastic limit is higher than at first, and almost as high as the load in the first test. Repeated overstrain of this kind, with subsequent annealing, makes it possible to raise the elastic limit considerably above what it was originally. When further strained the metal loses its elasticity and takes on a permanent **set**; that is to say, it does not return to its original length when the stress is removed. The elastic properties, however, can be restored by annealing (see Article 18). Overstrain in either tension or compression destroys almost entirely the elasticity of the material for strain of the opposite kind; for instance, a piece of mild steel overstrained in tension has its elastic properties in compression almost entirely destroyed, and vice versa. Overstraining in torsion produces much the same effect as overstraining in tension or compression.

181. Relative strength of large and small test pieces. It has been found by Tetmajer * and others that the values obtained in testing small test pieces taken from different parts of a steel girder or I-beam are higher than those obtained in testing the girder itself. The average of a series of tests of small test pieces gave an elastic limit of 49,000 lb./in.^{2} and a maximum strength of 62,000 lb./in.^{2} Tests on the complete girders themselves gave an elastic limit of 33,500 lb./in.^{2} and a maximum strength of 54,500 lb./in.^{2} The same has been found true for the elastic limit of wrought-iron girders, but in this case the maximum strength is greater in the girder than in the small test piece.

182. Strength of iron and steel at high temperatures. From a series of tests made at Cornell University,† it was found that wrought iron having a tensile strength of 30,000 lb./in.^{2} at ordinary temperatures increased in strength with increase of temperature up to 475° F., and then decreased as the temperature was further raised. Machinery steel of 60,000 lb./in.^{2} maximum strength gave at 475° F. a maximum strength of 111,500 lb./in.^{2} Tool steel having a strength of 114,000 lb./in.^{2} at ordinary temperatures gave 145,000 lb./in.^{2} maximum strength at 350° F.

Professor C. Bach also reports an elaborate series of tests on the strength of steel at high temperatures.‡ At ordinary temperatures one bar had a maximum strength of 54,000 lb./in.^{2}, an elongation in 8 in. of 26.3 per cent, and a contraction of area of 46.9 per cent. Up to a temperature of 572° F. the strength increased by about 7000 lb./in.^{2}, and from this point fell, approximately in proportion to the temperature, to 26,200 lb./in.^{2} at 1022° F. The ultimate elongation decreased to 7.7 per cent at 392° F., and then increased to 39.5 per cent at 1022° F. The contraction of area fell until 392° F. was reached, and did not rise until about 572° F.

While the tensile strength is increased for a moderately high temperature, the elastic limit is lowered in proportion to the increase of temperature, being diminished about 4 per cent for each increase of 100° F.

183. Character and appearance of the fracture. The kind and quality of the metal are usually indicated by the character of the

* *Communications*, Vol. IV. † *Journal Western Society of Engineers*, Vol. I.
‡ *Journal Franklin Institute*, December, 1904.

fractured portion of the test piece. Two points are to be noted in this connection: the **geometrical form** and the **appearance** of the fractured material. Under the first we may have, as in tensile tests of hard steel, a **straight fracture** where the material breaks squarely off in a plane at right angles to the axis of the test piece; or, as in tensile tests of mild steel and high-grade wrought iron, a fracture which is **cup-shaped**, **half-cup**, etc. The appearance of the material for the cup-shaped fracture may be described as *dull granular* in the bottom of the cup and *silky* around the edge; or, in the case of wrought iron, as *fibrous* in the bottom of the cup and silky around the edge. A cast-iron fracture appears *crystalline*, the crystals being *fine*, *coarse*, or *medium*.

In reporting a test the character and appearance of the fracture should always be given. It should also be noted whether or not any longitudinal seams occur, or whether the fracture shows the material to be homogeneous and free from blowholes and foreign matter. If the specimen has not been properly placed in the machine, so that there is a bending moment, the fracture will indicate this. The axis of the test piece should always coincide with the axis of the machine.

184. Measurement of extension, compression, and deflection. The extension in a tension specimen of iron or steel up to the elastic limit is so slight that very accurate measurements must be made to determine the elongations. Instruments for making such measurements are known as **extensometers**, and are usually made to read to .0001 of an inch. Fig. 173 shows a type of such instrument known as the *Yale-Riehlé extensometer*. The method of using the instrument is to mark off an 8-in. gauge length on the test piece and fasten the extensometer to it by inserting the screws in the extreme punch marks of the gauge length. The backpiece is then removed and a battery with a bell in circuit is attached; the instrument is then ready for use. As the piece elongates the elongations are measured by turning the micrometer screw until it touches the armature, when the circuit is closed and the bell rings.

The instrument is used only a little past the elastic limit (the limit of proportionality of stress to deformation), and about twenty elongations for corresponding loads are taken below the elastic limit. The instrument is then removed and the test continued to failure, the maximum load being noted. From the data obtained in making

the test, the strain diagram is drawn by using unit loads as ordinates and relative elongations as abscissas. From this curve the elastic limit, modulus of elasticity (Young's modulus), and modulus of elastic resilience may be determined.

The **elastic limit** is found by noting the point on the strain diagram where it ceases to be a straight line.

The **modulus of elasticity** is determined by dividing the stress by the deformation for any stress below the elastic limit.

The **modulus of elastic resilience** is defined as the amount of work required to deform a cubic inch of the material to its elastic limit. It is therefore represented by the area under the strain curve up to the elastic limit, or, expressed as a formula,

$$\textbf{Mod. elastic resilience} = \frac{\textbf{(stress at elastic limit)}^2}{\textbf{2 modulus of elasticity}}.$$

If in plotting the strain diagram the ordinates represent the stress expressed in lb./in.^{2} and the abscissas represent the corresponding unit elongations, the area under the curve up to the elastic limit multiplied by the scale value in inch-pounds of each unit area gives the modulus of elastic resilience in inch-pounds.

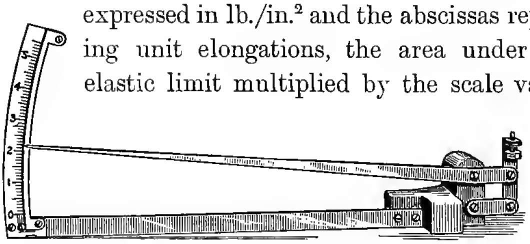


FIG. 172

The **modulus of total resilience** is defined as the amount of work required to deform a cubic inch of the material to rupture. It is therefore represented by the area under the whole curve multiplied by the scale value of a unit area, that is, the number of inch-pounds per unit area.

In case the stresses are plotted in pounds and the corresponding deformations in inches, the above method gives the work done on the whole volume of the specimen included in the gauge length. To obtain the modulus for such cases it is necessary, in addition to the above, to divide by the volume of that portion of the specimen over which the deformations were measured.

Compression is measured by means of a **compressometer**, by methods similar to those used in making tension tests. The strain diagram in this case is a *stress-compression* curve.

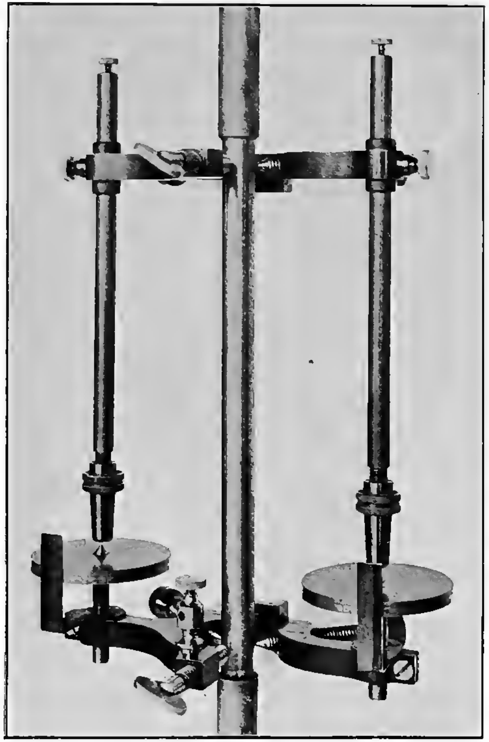


Fig. 173. — Extensometer

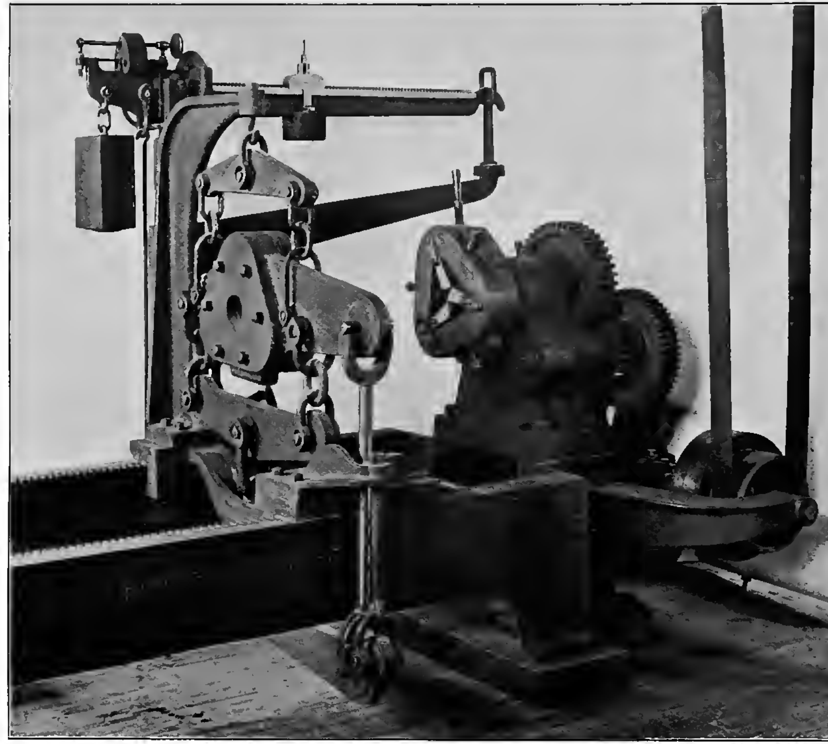


Fig. 174. — Torsion Testing Machine

For measuring deflections in transverse tests various methods are used. A simple instrument for this purpose is shown in Fig. 172. This instrument is placed under the beam and the deflections measured to .001 of an inch. The strain diagram for flexure is thus a *load-deflection* curve.

Problem 297. A rod of nickel steel .854 in. in diameter, and with a gauged length of 8 in., when tested in tension gave the data tabulated below. From this data draw the strain diagram and locate the elastic limit; also compute the modulus of elasticity and the modulus of elastic resilience.

Load lb./in.^{2}	Elongation inches per inch	Load lb./in.^{2}	Elongation inches per inch
4,000	.00018	36,000	.00096
8,000	.00032	40,000	.00103
12,000	.00040	44,000	.00115
16,000	.00050	48,000	.00125
20,000	.00060	52,000	.00165
24,000	.00065	56,000	.00470
28,000	.00075	91,400	Maximum load
32,000	.00083		

185. Torsion tests. The determination of the resistance of a material to shear or torsion is usually made by means of a machine designed to read twisting moment in inch-pounds on the scalebeam. The Riehlé machine shown in Fig. 174 may be taken as a type of torsion machines.

In making the test one end of the specimen is attached to the twisting head of the machine and the other end to the stationary head, which is connected by a system of levers to a scalebeam reading inch-pounds of moment. The machine shown in Fig. 174 has the stationary head suspended by stirrups, thus leaving it free to move slightly when the specimen shortens in twisting. The older types of torsion machine are not made to accommodate themselves in this way to the ~~shortening~~ of the test piece.

The angular distortion of the test bar is measured by an instrument called a **troptometer.** This consists of two arms attached to the bar at the extreme points of the part that is being tested. One of these arms carries a scale bent into the arc of a circle of which the arm is

the radius, and having its plane at right angles to the axis of the bar; the other arm carries a pointer so arranged as to move over the scale when the bar is twisted. The arc of the scale is called the *troptometer arc* and the arm supporting it the *troptometer arm.* The angular distortion at the center of the bar for the given gauge length is then obtained by dividing the reading on the troptometer arc by the length of the troptometer arm plus the radius of the specimen, or, expressed as a formula,

$$\text{Angle } \theta \text{ (in radians)} = \frac{\text{reading on troptometer arc}}{\text{troptometer arm} + \text{radius of specimen}},$$

where θ is the angle of twist (see Article 96).

Problem 298. A steel rod with a gauged length of 10 in. and .85 in. in diameter, when tested in torsion, gave the data tabulated below. Draw the strain diagram, plotting the stress in lb./in.^{2} on the outer fiber as ordinates, and the corresponding angle of twist θ as abscissas. Also locate the elastic limit, compute the modulus of elasticity of shear, and the modulus of elastic resilience. Length of troptometer arm, 12 inches.

TORSION TEST OF STEEL

Moment in. lb.	Troptometer Arc in.	Moment in. lb.	Troptometer Arc in.
0	0	1750	.33
250	.05	2000	.38
500	.10	2250	.43
750	.15	2500	.47
1000	.20	2750	.53
1250	.24	3000	.57
1500	.29		

186. Form of torsion test specimen. Specimens for torsion tests are made *cylindrical,* and usually long enough to get a gauged length of 10 in. The cylindrical form has been adopted because its cross sections remain plane during torsion, whereas in other forms a cross section which is plane before torsion is deformed into a warped surface by the strain, and therefore does not give a simple shearing stress (see Article 102). The torsion test is used to determine the shearing strength of materials, that is, the resistance offered by the material to one cross section slipping over another (see Article 68).

When torsion tests are made, the moment in in. lb. is read from the machine, and the shearing stress in the outer fiber in lb./in.^{2} is computed from the formula,

$$q = \frac{(Pa)r}{I_p},$$

where Pa is the twisting moment read from the machine, r the radius of the test piece, and I_p the polar moment of inertia of the cross section.*

The modulus of elasticity in shear is computed from the relation,

$$G = \frac{(Pa)l}{\theta I_p},$$

where Pa and I_p are defined as above, l is the gauged length in inches, and θ is the angle of twist in radians.

The test piece is held in position by a set of adjustable jaws similar to those used in ordinary pipe wrenches. The gauged length should be taken far enough from the ends so that the local stress due to the jaws may not influence the results.

187. Torsion as a test of shear. Although the torsion test is used to determine the shearing strength of materials, it is not an accurate test, since the shearing stress is a maximum on the outer elements, and zero at the center. For this reason the inner material tends to reënforce the outer, thus giving a higher shearing strength than would otherwise be obtained. A more perfect torsion test would be one made upon a hollow tube of the material, for in this case the inner reënforcing core would not be present. However, the difficulty of obtaining suitable hollow tubes for test pieces makes their use impracticable for ordinary tests.

A further objection to the torsion test as a test of shearing strength lies in the fact that there is considerable tension in the outer elements of the test piece during the test. Any element of the cylindrical test piece which is a straight line before the strain becomes a helix during the test. Since the length of the helix is greater than that of the original element, a tensile stress is thus produced in the outer fibers. In fact, in testing wrought iron in torsion the outer fibers often fail in tension along the helix. The slight shortening

* For a cylinder, $I_p = \frac{\pi r^4}{2}$.

There can be no such tension along helix lines until slip along helix lines has occurred, changing the piece into a rope of fibers. And the piece lengthens, instead of shortening; the diameter decreases

?

of the whole specimen, due to the twisting, is corrected in part by the swinging head of the machine shown in Fig. 174.

188. Shearing tests. To determine the shearing strength of timber along the grain and the resistance of iron and steel to the pulling out of rivets, many special tests are used. By means of a special piece of apparatus, the force required to push off, along the grain, a projecting piece from a test piece of timber is easily measured on the ordinary tension-compression machine. The intensity of shearing stress is computed by dividing the load by the area of the block pushed off.

of sheared faces

Tests are also made on wrought-iron plates to determine the force required to pull out a rivet through the metal, both in the direction of the fiber and perpendicular to it. A series of such tests may be found in the *Watertown Arsenal Report* for 1882. Many tests have also been made to determine the shearing strength of rivets.

189. Impact tests. In actual service many materials are subjected to shock or impact (see Article 74). This is especially true of all railway structural material, such as rails, axles, springs, couplers, bolsters, wheels, etc., which must be designed to withstand considerable shock. Two special machines have been designed to test materials in impact. The first, called the *drop testing* machine, is operated by allowing a given weight (hammer) to drop a given distance upon a test piece mounted on an anvil under the hammer. The other form of machine is operated by allowing a heavy pendulum to strike the specimen when placed in the center of its swing. In either case the amount of the energy of the blow absorbed by the specimen is desired.

The results obtained from impact tests can only be comparative in any case, since a part of the energy of the blow must be absorbed by the parts of the machine itself. This is seen in the drop testing machine in the absorption of energy by the anvil and hammer.

Since the results of such tests cannot be absolute, it is highly necessary that they should be standardized by making tests on the same anvil with the same hammer. The Master Car Builders Association has taken a step toward such standardization by building an impact testing machine for testing materials used by them. This machine has been established at Purdue University. Its maximum blow is given by a hammer having a weight of 1640 lb., and dropping 50 ft.

The use of this machine should do much to standardize specifications for railway material.*

Tests in impact compression, impact tension, and impact flexure are also made, but on account of the uncertainty as to the amount of energy absorbed by the test specimen many engineers do not favor such tests. Many of these objections, however, might be removed by proper standardization.

Some recent investigations seem to indicate that the impact test shows very little that cannot be determined by static tests.

190. Cold bending tests. Cold bending tests are tests of the ductility of metals, and are designed to show the effect on the metal of being bent in various ways while cold. Such material as rivet steel and Bessemer steel bridge pieces are bent double over a pin of specified radius, and the result noted. In making these tests the angle at which the first crack occurs and the angle at which rupture occurs are read.

Few machines for making cold bending tests have been made. The tests are usually made by bending the specimen over the edge of a vise, or some such simple device, according to specifications. The tests have never been standardized, but their importance is obvious, since the conditions of actual service are thus applied to the specimen.

191. Cast iron. Pig iron is a combination of iron with small percentages of carbon, silicon, sulphur, phosphorus, and manganese, obtained from the blast furnace. The carbon probably comes from the fuel used in reducing the ore; the other impurities come either from the ore or from the flux. The product is graded, according to chemical composition, into *forge pig* and *foundry pig*. Foundry pig is remelted in a cupola furnace and made into castings of various kinds; forge pig is used in making wrought iron.

Cast iron is a very brittle material, weak in tension and strong in compression. Its great usefulness in engineering structures comes from the fact that it may be readily molded into any desired form; it is, however, being replaced by the various steel products. The carbon, silicon, and other impurities contained in the iron affect its physical properties.

* For a description of this machine see the report by Professor W. F. M. Goss, *Proc. Amer. Soc. for Testing Materials*, 1903.

COMPOSITION AND TENSILE STRENGTH OF CAST IRON

WATERTOWN ARSENAL REPORT, 1895

Chemical Composition						Tensile Strength
Carbon		Manganese	Silicon	Sulphur	Phosphorus	lb./in.^{2}
Graphitic	Combined					
2.917	.570	.464	1.457	.122	.539	28,980
2.367	.529	.458	1.022	.120	.379	28,620
2.017	.710	.438	1.193	.125	.350	27,240
2.691	.394	.439	1.494	.104	.538	24,970
2.786	.794	.446	1.372	.100	.521	24,880
2.765	.589	.437	1.457	.093	.537	29,720
2.140	1.132	.445	0.705	.104	.504	32,020
2.372	1.006	.436	0.921	.105	.473	33,500
2.356	0.952	.432	1.071	.100	.531	33,400
2.263	1.009	.451	0.846	.108	.505	32,010
2.247	0.867	.441	0.977	.110	.472	32,990
2.160	1.068	.435	0.864	.095	.493	32,280
2.208	0.982	.430	0.874	.096	.498	32,200
2.266	1.180	.426	0.893	.102	.504	30,400
2.225	1.074	.451	0.902	.095	.473	32,510

Carbon occurs as *combined carbon* or as *graphitic carbon.* Combined carbon makes the metal hard, brittle, white, weak in tension, and strong in compression, whereas graphitic carbon makes the iron soft, gray, and weak in both tension and compression. Graphitic carbon occurs in the metal as a foreign substance, which probably accounts for its weakening effect. *Silicon* in cast iron up to 0.5 per cent increases its compressive strength. The tensile strength is increased up to 2 per cent. *Manganese* as it usually occurs is not injurious below 1 per cent. When more is present the shrinkage, hardness, and brittleness are rapidly increased. *Phosphorus* makes the iron weaker and less stiff, becoming a serious impurity when it occurs in quantities above 1.5 per cent. *Sulphur* causes whiteness, brittleness, hardness, and greater shrinkage, and is, in general, a very objectionable impurity.

Cast iron has an average tensile strength of 22,500 lb./in.^{2}, the range being from 13,000 lb./in.^{2} to 35,000 lb./in.^{2} Its compressive

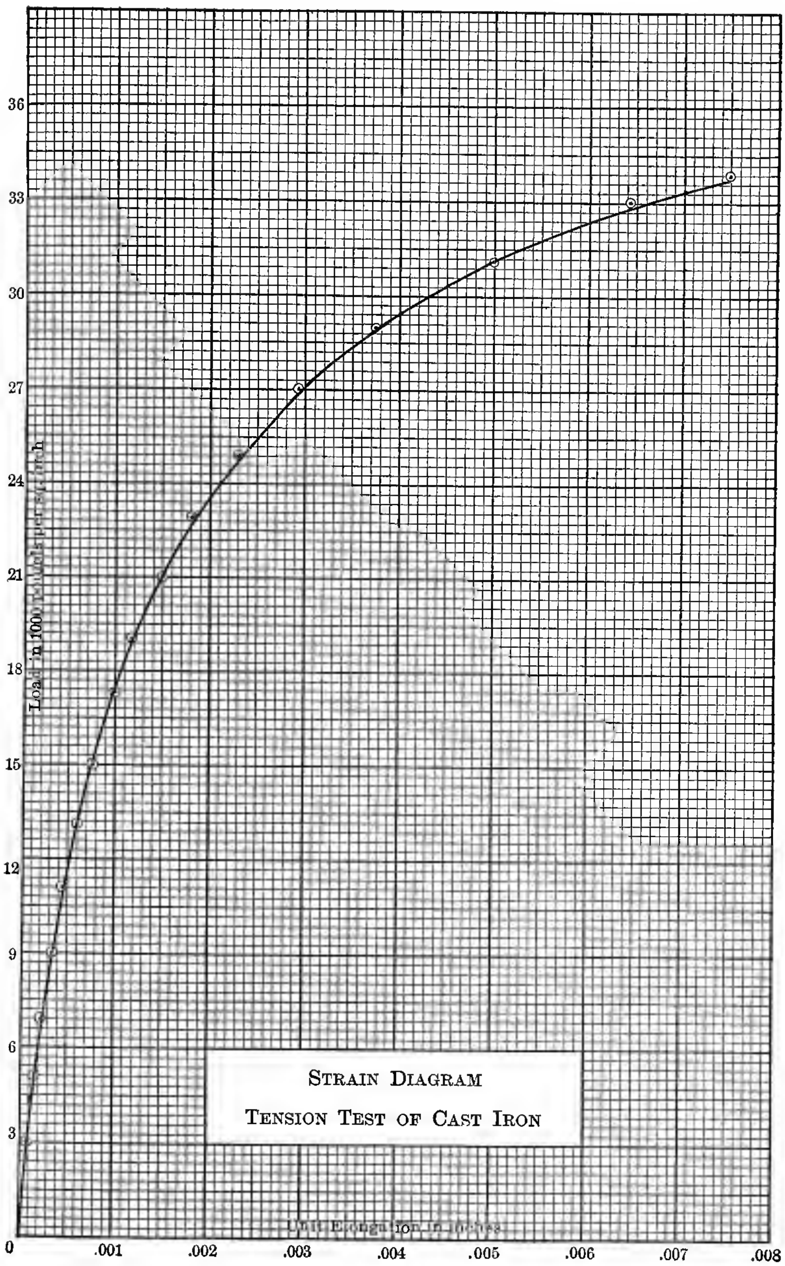

Strain Diagram
Tension Test of Cast Iron
Load in 1000 pounds per sq. inch
Unit Elongation in inches
36
33
30
27
24
21
18
15
12
9
6
3
0
.001
.002
.003
.004
.005
.006
.007
.008

Fig. 175

strength varies from 50,000 lb./in.^{2} to 150,000 lb./in.^{2}, a good average being about 95,000 lb./in.^{2}

The metal is so imperfectly elastic that Hooke's law does not strictly hold for any range of stress, however small. The modulus of elasticity in tension varies from 15,000,000 to 20,000,000 lb./in.^{2}, and in shear from 5,000,000 to 7,000,000 lb./in.^{2} On page 280 is given a table of the tensile strength of various samples of cast iron of different chemical compositions.

192. Strain diagram for cast iron. The strain diagram of cast iron in tension, shown in Fig. 175, illustrates clearly the fact that the metal is very imperfectly elastic. No part of the diagram is a straight line, and no elastic limit is shown by the curve. The maximum load in this case was 34,750 lb./in.^{2} The curve was drawn from data given in the *Watertown Arsenal Report*, 1895. From the results of four hundred and fifty tests of cast iron in tension, compression, and cross-bending, Kirkaldy found the average compressive strength to be 121,000 lb./in.^{2}, the tensile strength 25,000 lb./in.^{2}, and the cross-bending modulus (see Article 65) 38,000 lb./in.^{2}

Fig. 176 shows a strain diagram of cast iron in compression. Like the tension diagram, this shows no well-defined elastic limit and no constant modulus of elasticity. The maximum compressive strength in this case was 50,000 lb./in.^{2}

When tested in compression as a short block, cast iron has a characteristic fracture, shearing along a plane making an angle of about 35° with the vertical. This differs by 15° from the theoretical angle (45°) of maximum stress for such cases.

193. Cast iron in flexure. The most extended series of tests ever made on cast iron in flexure was made by J. W. Keep on bars $\frac{1}{2}$ in. square and 12 in. long. From these tests, the average strength was found to be 450 lb., giving a modulus of rupture of 64,800 lb./in.^{2} A good average for the modulus of rupture for ordinary commercial cast iron would be between 36,000 lb./in.^{2} and 42,000 lb./in.^{2}

194. Cast iron in shear. The strength of cast iron in shear varies from 13,000 lb./in.^{2} to 25,000 lb./in.^{2} Tests are made in the ordinary torsion machine. The fracture in this case is the characteristic fracture of brittle materials in torsion; that is, instead of shearing off in a plane at right angles to the axis of the test piece, as is the case with

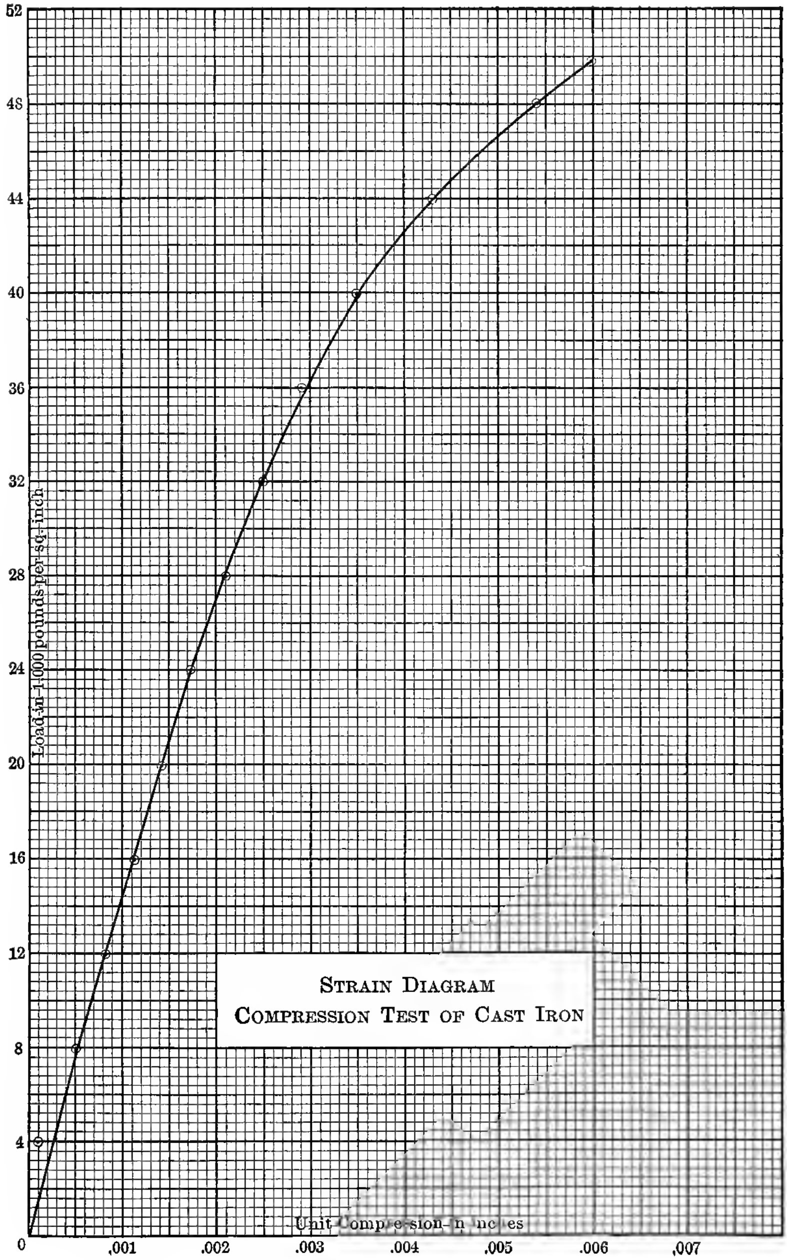

Strain Diagram
Compression Test of Cast Iron
Load in 1,000 pounds per sq. inch
Unit Compression in Inches
52
48
44
40
36
32
28
24
20
16
12
8
4
0
.001
.002
.003
.004
.005
.006
.007

Fig. 176

ductile materials, the fracture extends down one side for some distance. The material fails by the outer fiber failing first in tension. A similar fracture can be seen by twisting a stick of chalk or other brittle material with the fingers until fracture occurs.

195. Cast-iron columns. Some tests have been made upon full-sized cast-iron columns both at the Watertown Arsenal and by the Phœnix Iron Company of Phœnixville, Pennsylvania. The results of these tests show that the total strength of these columns is much less than the compressive strength of the metal would lead one to expect. This was probably due to the presence of blowholes or other imperfections in the column, such as are likely to occur when large pieces are cast. The ultimate strength of the Watertown columns varied from 21,000 lb./in.^{2} to 40,000 lb./in.^{2}

The following table gives the result of a compression test of a cast-iron column made by the Watertown Arsenal, the ultimate strength in this case being 33,340 lb./in.^{2}

COMPRESSION TEST OF CAST-IRON COLUMN

Gauge length, 100 in. Sectional area, 17 in.^{2}

WATERTOWN ARSENAL REPORT, 1893

LOAD lb./in.^{2}	COMPRESSION in.	DEFLECTION AT MIDDLE in.	LOAD lb./in.^{2}	COMPRESSION in.	DEFLECTION AT MIDDLE in.	
0	0	0	18,000	.1390	.05	
500	0	0	20,000	.1597	.06	
1,000	.0032	0	22,000	.1816	.08	
2,000	.0093	0	24,000	.2080	.10	
4,000	.0225	0	26,000	.2430	.12	
6,000	.0373	.01	28,000	. . .	.17	
8,000	.0530	.02	30,000	. .	.24	
10,000	.0688	.02	32,000	. .	.40	
12,000	.0853	.02	33,000	. . .	.66	
14,000	.1023	.03	33,340	. . .	1.10	Ultimate strength
16,000	.1204	.04	. . .	. .		

Problem **299**. The data in the preceding table were obtained from a round, hollow, cast-iron column 120 in. in length, 3.05 in. in external diameter, and 1.97 in. in internal diameter. Draw the load-compression and load-deflection

curves for this case, and determine whether or not an elastic limit is indicated. Also compute the strength of the column by Rankine's formula and Johnson's straight-line formula, and compare the results with those obtained from the test.

196. Malleable castings. The castings with combined carbon are hard and brittle. These are heated with some oxide, so that the carbon near the surface is burned out, leaving the outer surface tough and strong, like wrought iron. The interior of the casting is somewhat annealed, but the finished product consists of a hard interior portion with a ductile outer portion. This structure insures strength both statically and as regards impact.

197. Specifications for cast iron.* The following specifications are for *special hard cast iron* (close-grained). They are taken from the J. I. Case Threshing Machine Company's specifications, and may be considered as typical.

Chemical Composition

Silicon must be between 1.20 and 1.60 per cent. (Below 1.20 the metal will be too hard to machine; above 1.60 it is likely to be porous unless much scrap is used.)

Sulphur must not exceed 0.095 per cent, and any casting showing on analysis 0.115 per cent or more of sulphur will cause the rejection of the entire mixture. (Above 0.115 per cent sulphur produces much shrinkage, shortness, and "brittle hard" iron.)

Phosphorus should be kept below .70 per cent unless specified for special thin castings. (High phosphorus gives castings brittle under impact.)

Manganese should not be above .70 per cent except in special chilled work.

Physical Tests

Transverse breaking strength. The test bars should be 1 in. square and $13\frac{1}{3}$ in. long, and should be tested with a load of 2400 lb. applied at the center of a 12-in. span.

* These specifications, as well as all others quoted, are given so that the student may get an idea of the composition and properties required of commerial cast iron or other material. Specifications issued by different companies vary, and those issued by the same company are frequently changed on account of the requirements of service.

Deflection should not be less than 0.08 in.

Tensile strength must not be less than 22,000 lb./in.^{2}

The following specifications for cast iron are suggested by J. W. Keep as being representative of modern practice.*

Transverse test bars were cast 1 in. square and 12 in. long, and were tested with a central load. Tensile test bars were cast 1.13 in. in diameter and were tested as cast.

Character of Casting		Silicon Range	Sulphur below	Phosphorus below	Manganese below	Transverse Strength lb.	Tensile Strength lb./in.^{2}
Furnace	Heavy	1.20–1.50	.085	.24	.37	3900	38,000
Furnace	Medium	1.50–2.00	.085	.68	.04	3200	31,000
Cupola	Special, Heavy	1.20–1.50	.090	.60	.80	2600	25,000
Cupola	Special, Medium	1.50–2.00	.080	.60	.80	2400	23,000
Cupola	General, Heavy	1.20–1.75	.090	.70	.70	2400	22,000
Cupola	General, Medium	1.40–2.00	.085	.70	.70	2200	20,000
Cupola	General, Light	2.20–2.80	.085	.70	.70	2000	18,000
Cupola	Chemical Work	1.10–1.35	.070	.25	.60	....	....
Cupola	Brake Shoes	2.00–2.50	.150	.70	.70	2900	28,000
Chilling Iron		Below 1.00	..	..	..	...	...

198. Wrought iron and steel. Wrought iron is made by burning the impurities out of cast iron. In the process the foundry pig iron from the blast furnace is first placed in the puddle furnace, where it is heated and stirred until the carbon, silicon, and manganese are almost entirely burned out. When taken from the furnace, the iron is in the form of a pasty ball, which is squeezed until the cinders are expelled, after which it is rolled into bars known as *muck bars.* After being reheated it is rolled again, and is then known as *merchant bar.* If a better grade of wrought iron is desired, the merchant bar is reheated and rolled again, when it is known as *best iron;* if rolled again, the quality is still further improved.

In methods used by the ancients the ore and fuel were placed together. This necessitated a pure fuel and did not admit of rapid manipulation; it is still used, however, to obtain wrought iron of a pure quality, and in obtaining very fine grades of steel.

* *Proc. Amer. Soc. for Testing Materials,* 1904.

Wrought iron is a tough, ductile material showing an elongation of from 18 to 30 per cent in 8 in. Its tensile and compressive strength at the elastic limit is about 28,000 lb./in.^{2} for high-grade wrought iron, and about 23,000 lb./in.^{2} for commou wrought iron. Its maximum tensile strength varies from 44,000 lb./in.^{2} to 64,000 lb./in.^{2} The material is much more elastic than cast iron, its modulus of elasticity in tension being about 28,000,000 lb./in.^{2}, and in shear about 10,000,000 lb./in.^{2}

Ingot iron. The impurities of wrought iron have almost been eliminated in a new product known as ingot iron. In the manufacture of this material the carbon, manganese, sulphur, and phosphorus are nearly all burned out, leaving the product 99.94 per cent pure iron, which greatly increases its strength and ductility (see *Ry. Age Gazette*, Vol. 49, p. 574). It does not corrode easily, and has good electrical conductivity and low magnetic retentivity.

199. Manufacture of steel. Tool steel is made by recarbonizing wrought iron by heating it in a charcoal fire for several days at a temperature of about 3000° F. During this process part of the carbon is absorbed by the iron, the product being known as *blister steel.* This is then melted and cast into ingots, from which the merchantable bars are rolled or hammered. The two steps in this process are usually combined into one.

Tool steel. Carbon tool steel, such as has been used until within the past few years, did not admit of high speeds when cutting. The rubbing of the chip soon dulled the tool, and any considerable increase in temperature was sufficient to cause it to lose its hardness. It has been unusual for such steel to stand a speed of cutting of 50 ft./min.

The constituents of carbon steel as previously used for cutting tools are indicated by the following table (see Becker, *High-Speed Steel*):

Use	Iron	Manganese	Silicon	Sulphur	Phosphorus	Carbon
Hammers	99.040	.21	.21	.022	.020	.50 to .75
Knives	98.935	.20	.18	.020	.015	.65 to .80
Drills, reamers . . .	98.731	.18	.21	.015	.014	.85 to 1.30
Lathe tools . . .	98.520	.26	.20	.010	.010	1.00 to 1.30
Razors	98.265	.22	.20	.006	.009	1.30 to 1.50
Carving tools . . .	98.374	.16	.14	.014	.012	1.30 to 1.50

High-speed steel. Recently it has been found that the addition of tungsten and other constituents had the effect of so changing the tool steel as to increase its wearing qualities and to make it capable of cutting at a much higher speed than formerly. A speed of 500 ft. per minute is often obtained with this new steel, although the average is considerably less than this. The tool may be heated up to redness in cutting without injuring its wearing qualities appreciably. This high-speed steel, as it is called, has made very rapid work possible.

The chemical analysis of twenty brands of this material is given by Becker as follows:

	AVERAGE	HIGH	LOW
Carbon	.75	1.28	.32
Tungsten	18.00	25.45	14.23
Molybdenum	3.50	7.6	0.00
Chromium	4.00	7.2	2.23
Vanadium	.30	.32	0.00
Manganese	.13	.30	.03
Silicon	.22	1.34	.43
Phosphorus	.018	.029	.013
Sulphur	.010	.016	.008

Open-hearth steel is obtained by mixing molten pig iron with scrap iron or scrap steel in an open-hearth furnace. The ~~added scrap~~ is low in carbon, and thus lowers the percentage of carbon in the mixture. To offset this, the desired amount of carbon is introduced by adding *spiegeleisen.*

Bessemer steel is made directly from pig iron in a Bessemer converter, no additional fuel other than the impurities in the metal being used. These impurities are burned out to the desired extent by forcing jets of hot air through the liquid metal. Since in this method the molten iron is taken directly from the blast furnace, a considerable saving in the cost of production is effected, by reason of which the Bessemer process has revolutionized the steel industry.

In both the open-hearth and Bessemer processes the liquid steel is cast into ingots, which are rolled into the desired shapes.

200. Composition of steel. The physical properties of steel are largely modified by the relative proportions in which the various ingredients are present.

Carbon. Increasing the amount of carbon in steel has, in general, the effect of increasing its modulus of elasticity and its ultimate strength. From a series of tests made on carbon steel, in which the percentage of carbon varied from 0.08 to 1.47, Professor Arnold found that the elastic limit varied from 27,300 lb./in.^{2} to 72,300 lb./in.^{2}; the tensile strength, from 47,900 lb./in.^{2} to 124,800 lb./in.^{2}; the elongation, from 46.6 per cent to 2.80 per cent; and the reduction of area, from 74.8 per cent to 3.30 per cent.* The following table gives average values of the ultimate strength in both tension and compression for Bessemer and open-hearth steel containing different percentages of carbon.

Per Cent of Carbon	Tensile Stress		Compressive Stress	Shearing Stress
	Elastic Limit lb./in.^{2}	Maximum lb./in.^{2}	Elastic Limit lb./in.^{2}	Maximum lb./in.^{2}
0.15	42,000	64,000	40,000	48,000
0.20	46,000	70,000	43,000	54,000
0.50	50,000	78,000	48,000	56,000
0.70	54,000	88,000	54,000	59,000
0.80	58,000	99,000	62,000	68,000
0.96	70,000	115,000	70,000	80,000

Carbon tool steel furnishes material for springs, saws, chisels, files, etc. When annealed it is strong in both tension and compression, and quite ductile, but when heated to the critical temperature and then quenched, it becomes weak, brittle, and hard.

Silicon in carbon steel and wrought iron generally strengthens the material, but decreases its ductility. In amount it is usually less than 0.6 per cent.

Manganese increases both the strength and hardness of carbon steel and wrought iron, and decreases ductility to some extent. More than 1.5 per cent makes the steel very brittle. When manganese is present in quantities of from 10 to 35 per cent, with a small amount of carbon, say 1 per cent, the steel becomes hard and is used for castings and forgings. When annealed the castings are both strong and tough enough to resist wear. Rolled manganese steel is also produced.

* *Proc. Inst. Civ. Eng.*, 1895.

Manganese steel. This is coming into use for railroad rails on account of its resistance to wear. The average values for the strength of this material may be given as follows:

	Elastic Limit lb./in.^{2}	Tensile Strength lb./in.^{2}
Cast manganese steel	45,000	82,000
Rolled manganese steel	60,000	135,000

Sulphur increases the brittleness and hardness of steel and wrought iron, and is, in general, a very harmful ingredient. Low percentages of sulphur somewhat increase the tensile strength.

Phosphorus increases hardness and tensile strength, but decreases ductility, making the metal weak under impact and unsuited for anything but static loads.

Nickel is added to steel up to about 35 per cent. When the percentage of nickel is low, say about 5 per cent or less, the elastic limit and tensile strength are raised without any reduction in the elongation or in the contraction of area. Because of this increase in strength without loss of ductility, nickel steel is used in the manufacture of armor plate, armor-piercing shells, boiler tubes, shafting, etc., where a steel is needed which shall combine great strength with toughness. The following table shows the relative properties of low carbon steel tubes and high nickel steel tubes.*

Properties	Low Carbon Steel Tubes	High Nickel Steel Tubes
Tensile strength, lb./in.^{2} . .	50,000–55,000	85,000–95,000
Elastic limit, lb./in.^{2} . . .	30,000–35,000	40,000–45,000
Per cent of elongation in 8 in.	20–30	20–30

The same authority also gives the average tensile strength of sixteen steel tubes, composed of 25 per cent nickel, as 108,913 lb./in.^{2} for unannealed specimens, and 97,300 lb./in.^{2} for annealed specimens. The elongation in the former case was 28 per cent in 7.87 in., and in the latter case 38 per cent in the same length.

* *Proc. Soc. Naval Architects and Marine Engineers,* November, 1903.

Tests made by the Watertown Arsenal on a 3.37 per cent nickel steel gave an average elastic limit of 56,700 lb./in.^{2} and a tensile strength of 90,300 lb./in.^{2} *

Vanadium steel. Vanadium when added to steel in small quantities acts as a dynamic intensifier, that is to say, it greatly increases resistance to fatigue under alternating or repeated stresses. Vanadium also increases the static properties of steel, increasing its strength and toughness and resistance to wear or abrasion. Tensile tests made by the writer on one grade gave the following results: Elastic limit, 78,000 lb./in.^{2}; yield point, 91,900 lb./in.^{2}; tensile strength, 116,100 lb./in.^{2}; modulus of elasticity, 30,900,000 lb./in.^{2}; elongation in 8 in., 20 per cent; reduction of area at fracture, 57 per cent. This material showed the following chemical analysis: Vanadium, .30; carbon, .25; manganese, .15; chrome, .42; phosphorus, .009; sulphur, .024; silicon, .10.

Recent tensile tests of large I-bars of vanadium steel having a cross section of 14 in. × 2 in. gave an average elastic limit of 70,000 lb./in.^{2}, an average yield point of 81,200 lb./in.^{2}, and an average maximum strength of 96,800 lb./in.^{2} (see *Eng. Record,* July 30, 1910). The following chemical analysis is given for this steel: carbon, .25; vanadium, .17; nickel, 1.45; chrome, 1.20; manganese, .32; silicon, .12; phosphorus, .02; sulphur, .035. In engineering steels the maximum amount required seldom exceeds 0.2 per cent. Its judicious use makes it possible to fulfill varied requirements, whether chiefly static, chiefly dynamic, or divided between the two.

201. Steel castings are made both by the Bessemer and open-hearth processes. In the Bessemer process the iron is first reduced to wrought iron, and then *spiegeleisen,* or ferromanganese, added to furnish the necessary carbon. Aluminum may be added to prevent blowholes. The metal is cast in the same way as in making other castings.

On page 292 is given a report of a series of tests made at the Watertown Arsenal on castings for gun carriages.† The elastic limit varied from 47,000 lb./in.^{2} to 21,500 lb./in.^{2}, and the tensile strength from 81,000 lb./in.^{2} to 43,000 lb./in.^{2} Good average values might be given as 30,000 lb./in.^{2} at the elastic limit and 66,000 lb./in.^{2} at the

* *Watertown Arsenal Report,* 1899.

† *Watertown Arsenal Report,* 1903.

maximum. At the elastic limit the compressive strength was about the same as the tensile strength. The American Society for Testing

TEST OF STEEL CASTINGS

Elastic Limit lb./in.^{2}	Tensile Strength lb./in.^{2}	Elongation in 2 in. per cent	Contraction of Area per cent	Appearance of Fracture
43,500	81,500	24.5	49.1	Fine silky
43,500	78,600	28.5	49.1	"
44,000	80,500	26.5	46.2	"
47,000	73,000	28.5	59.8	"
44,500	73,500	28.5	57.2	"
42,000	78,550	26.5	51.9	"
46,000	74,800	32.0	59.8	"
46,500	73,600	28.5	57.2	"
43,750	75,200	30.0	57.2	"
42,500	75,100	28.5	57.2	"
34,000	67,000	32.0	51.9	Silky
42,000	67,100	32.0	51.9	"
38,500	67,700	27.5	40.2	"
26,000	43,000	8.5	20.5	Dull silky; granular spots; blowhole
25,980	60,620	29.0	42.2	Dull silky, 80 per cent; granular, 20 per cent
24,960	60,370	23.5	29.4	Granular, silvery luster, 90 per cent; dull silky, 10 per cent
30,060	60,110	29.0	48.1	Dull silky
32,500	68,000	19.5	40.3	Granular, 50 per cent; dull silky, 50 per cent
31,500	66,750	29.5	40.3	Dull silky
29,000	66,500	24.0	40.3	"
24,450	60,620	27.0	36.0	Dull silky, 40 per cent; granular, 60 per cent
27,000	60,750	6.5	16.9	Dull silky; blowhole
26,000	66,250	12.5	23.9	Granular, silvery luster, 85 per cent; dull silky, 15 per cent
22,500	58,500	31.0	40.3	Granular, silvery luster, 60 per cent; dull silky, 40 per cent
26,500	66,750	20.0	43.3	Dull silky
38,000	66,250	22.5	37.1	Dull silky, 60 per cent; granular, 40 per cent
26,000	69,000	9.0	27.4	Dull silky; granular spots
24,450	59,600	25.5	32.8	Dull silky, 80 per cent; granular, 20 per cent
27,500	60,500	27.0	37.1	Dull silky; trace of granulation
25,500	63,750	28.5	37.1	Dull silky; granular spots
31,000	66,750	16.5	16.9	Granular, silvery luster, 80 per cent; dull silky, 20 per cent
37,500	68,000	14.5	16.9	Granular, silvery luster, 85 per cent; dull silky, 15 per cent
26,000	59,000	21.5	37.1	Dull silky; granular spots
27,500	60,500	21.5	40.3	Dull silky, 90 per cent; granular, 10 per cent
24,960	60,880	26.5	36.0	Dull silky
26,490	64,700	17.0	22.5	Granular, silvery luster
25,470	63,930	15.5	19.0	"
29,040	65,210	29.0	48.1	Dull silky
26,500	64,500	30.0	46.2	"
27,500	67,750	15.0	16.9	Dull silky, 50 per cent; granular, 50 per cent
27,000	64,750	16.0	16.9	Dull silky, 20 per cent; granular, 80 per cent

Materials has recommended the following values for the strength of steel castings (allowable variation 5000 pounds).

	Tensile Strength lb./in.^{2}
Soft castings	60,000
Medium castings	70,000
Hard castings	80,000

In the cold bending test the material must be bent about a diameter of 1 in. through 120° for the soft, and 90° for the medium, without showing cracks or signs of failure.*

The Ordnance Department of the United States Army in the general specifications for 1903 gives the following requirements for steel castings and forgings.

Metal	Elastic Limit lb./in.^{2}	Tensile Strength lb./in.^{2}	Elongation after Rupture per cent	Contraction of Area per cent
Cast steel, No. 1	25,000	60,000	18.0	27.0
	28,000	65,000	16.0	24.0
Cast steel, No. 2	35,000	75,000	15.0	20.0
Cast steel, No. 3	45,000	85,000	12.0	18.0
Forged steel, No. 1	27,000	60,000	28.0	40.0
Forged steel, No. 2	35,000	75,000	20.0	30.0
Forged steel, No. 3	42,000	90,000	16.0	24.0

202. Modulus of elasticity of steel and wrought iron. The modulus of elasticity of steel and wrought iron is about the same in tension as in compression. For steel, 30,000,000 lb./in.^{2} is usually taken as a good average value for tension and compression, and about two fifths of this amount, or from 10,000,000 to 12,000,000 lb./in.^{2}, for shear; for loads below the elastic limit it is always the ratio of stress to deformation.

From a series of tests reported in the *Trans. Amer. Soc. Civ. Eng.*, Vol. XVII, pp. 62–63, the following average values are found.

Material	Modulus of Elasticity lb./in.^{2}	
	Tension	Compression
Ordinary steel	30,000,000	29,000,000
Spring steel	29,500,000	29,300,000
Wrought iron	28,200,000	27,600,000

* *Proc. Amer. Soc. for Testing Materials*, 1903.

203. Standard form of test specimens. It was pointed out in Article 20 that the form of the test specimen had considerable effect upon the results obtained from tests. To eliminate this factor, standard

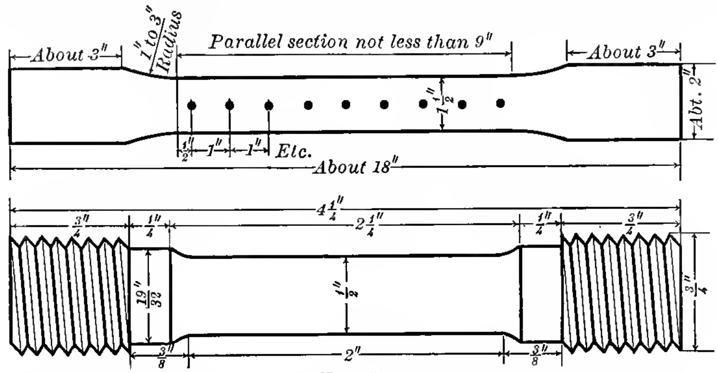


FIG. 177

dimensions for both cylindrical and rectangular test specimens have been adopted. These are shown in Fig. 177.

204. Specifications for wrought iron and steel. In order that the student may form some idea of the strength required by manufacturers for different grades of wrought iron and steel, quotations are given below from the specifications of the American Society for Testing Materials.

WROUGHT IRON

	Stay-Bolt Iron	Merchant Grade A	Merchant Grade B	Merchant Grade C
Tensile strength, lb./in.^{2} . .	46,000	50,000	48,000	48,000
Yield point, lb./in.^{2} . .	25,000	25,000	25,000	25,000
Per cent of elongation in 8 in.	28	25	20	20

STEEL

	Rivet Steel	Soft Steel	Medium Steel
Tensile strength, lb./in.^{2} . .	50,000–60,000	52,000–62,000	60,000–70,000
Yield point, lb./in.^{2}	30,000	32,000	35,000
Elongation in per cent for 8 in. shall not be less than .	26	25	22

The above grades of steel, known as structural steel for bridges and ships, must conform to certain bending tests. For this purpose the test specimens shall be $1\frac{1}{2}$ in. wide, if possible, and for all material $\frac{3}{4}$ in. or less in thickness the test specimen shall be of the same thickness as that of the finished material from which it is cut; but for material more than $\frac{3}{4}$ in. thick the bending test specimen may be $\frac{1}{2}$ in. thick. Rivet rounds shall be tested full size as rolled.

Rivet steel shall bend cold 180° flat on itself without fracture on the outside of the bent portion.

Soft steel shall bend cold 180° flat on itself without fracture on the outside of the bent portion.

Medium steel shall bend cold 180° around a diameter equal to the thickness of the specimen tested, without fracture on the outside of the bent portion.

STEEL AXLES

Steel for axles shall be made by the open-hearth process and shall be divided into the following classes: (*a*) car, engine-truck, and tender-truck axles; and (*b*) driving axles. For (*a*) no tensile tests shall be required, but for driving axles the following physical properties shall be required.

	Carbon Steel	Nickel Steel
Tensile strength, lb./in.^{2}	80,000	80,000
Yield point, lb./in.^{2}	40,000	50,000
Contraction of area in per cent	. . . . . .	45
Per cent of elongation in 2 in.	20	25

The same specifications require that one axle taken from each melt shall be tested by the drop test, as follows.

Diameter of Axle at Center in.	Number of Blows	Height of Drop ft.	Deflection in.
$4\frac{1}{4}$	5	24	$8\frac{1}{4}$
$4\frac{3}{8}$	5	26	$8\frac{1}{4}$
$4\frac{7}{16}$	5	$28\frac{1}{2}$	$8\frac{1}{4}$
$4\frac{5}{8}$	5	31	8
$4\frac{3}{4}$	5	34	8
$5\frac{3}{8}$	5	43	7
$5\frac{7}{8}$	7	43	$5\frac{1}{2}$

To be accepted, the axle must stand the blow without rupture and without exceeding, as the result of the first blow, the deflection stated.

DESCRIPTION OF THE DROP TEST

The points of support on which the axle rests during tests shall be 3 ft. apart from center to center; the hammer must weigh 1640 lb.; the anvil, which is supported on springs, must weigh 17,500 lb.; it must be free to move in a vertical direction; the springs upon which it rests must be twelve in number, of the kind specified; and the radius of supports and of the striking face on the hammer in the direction of the axis of the axle must be 5 in.

The deflections are measured by placing a straightedge along the axle, properly held at the supports, and measuring the distance from this straightedge to the axle both before and after the blow. The difference between the two measurements gives the deflection.

CHAPTER XIII

LIME, CEMENT, AND CONCRETE

205. Quicklime. If calcium carbonate (ordinary limestone) is heated to about 800° F., carbon dioxide is driven off, leaving an oxide of calcium, which is known as **quicklime**. This has a great affinity for water and *slacks* upon exposure to moisture. Slacked lime when dry falls into a fine powder.

Lime mortar is formed by mixing slacked lime with a large proportion of sand. Upon exposure to the air this mortar becomes hard by reason of the lime combining with carbon dioxide and forming again calcium carbonate, the product being a sandy limestone. Lime mortar is used in laying brick walls and in structures where the mortar will not be exposed to water, since it will not set, i.e. combine with carbon dioxide, under water.

206. Cement. When limestone contains a considerable amount of clay, the lime produced is called **hydraulic lime**, for the reason that mortar made by using it will harden under water. If the limestone contains about 30 per cent of clay and is heated to 1000° F., the carbon dioxide is driven off, and the resulting product, when finely ground, is called **natural cement**. When about 25 per cent of water is added, this cement hardens, because of the formation of crystals of calcium and aluminum compounds.

If limestone and clay are mixed in the proper proportions, usually about three parts of lime carbonate to one of clay, and the mixture roasted to a clinker by raising it to a temperature approaching 3000 F., the product, when ground to a fine powder, is known as **Portland cement**. The proper proportion of limestone and clay is determined by finding the proportions of the particular clay and stone that will make perfect crystallization possible. In the case of natural cement the lime and clay are not present in such proportions as to form perfect crystals, and consequently it is not as strong as Portland cement.

The artificial mixing of the limestone and clay in the manufacture of Portland cement is accomplished in different ways. Throughout the north central portion of the United States large beds of *marl* are found, and also in the same localities beds of suitable clay. This marl is nearly pure limestone, and is mixed with the clay wet. (These materials are also mixed dry.) Both the marl and clay are pumped to the mixer, where they are mixed in the proper proportions. The product is then dried, roasted, and ground. Most American Portland cements, however, are made by grinding a clay-bearing limestone with sufficient pure limestone to give the proper proportions. After being thoroughly mixed the product is roasted and ground to a powder.

Slag cement (Puzzolan) is made by thoroughly mixing the granulated slag from an iron blast furnace with slacked lime, and then grinding the mixture to a fine powder. Slag cements are usually lighter in color than the Portland cements, and have a lower specific gravity, the latter ranging from 2.7 to 2.8. They are also somewhat slower in setting than the Portland cements, and have a slightly lower tensile strength. They are not adapted to resist mechanical wear, such as would be necessary in pavements and floors, but are suitable for foundations or any work not exposed to dry air or great strain.

True Portland cement may be made from a mixture of blast-furnace slag and finely powdered limestone, the mixture being burned in a kiln and the resultant clinker ground to powder. Both the Portland and the Puzzolan cements will set under water, i.e. they are hydraulic.

207. Cement tests. The many different processes of mixing, roasting, grinding, and setting through which a cement must pass, require that a number of tests be made to determine whether or not these have been well done. If the grinding has been improperly done, or if any of the other operations of manufacture have been neglected, the product may be very weak, or even worthless. To make sure that all the steps in the manufacture of the cement have been properly carried out, engineers make use of the following tests: (*a*) test of soundness; (*b*) test of fineness; (*c*) test of time of setting; (*d*) test of tensile strength.

208. Test of soundness. One test for soundness consists in boiling a small ball of neat cement in water for three hours, and noting whether or not checks or cracks occur. If the cement contains too

much free lime, the ball will disintegrate and show signs of crumbling. The ball of cement is kept under a damp cloth for twenty-four hours before boiling. This test is not regarded with favor by many engineers (see steam pat test, specifications, p. 305).

209. Test of fineness. If the grinding has not been properly done, large particles of clinker remain, which act as a sand or other foreign substance and thus weaken the cement. The test for fineness is made by sifting the cement through different sieves; usually all of it is required to pass through a sieve of 50 meshes to the inch, and a smaller amount through sieves of 80 and 100 meshes. About 75 per cent should pass through a 200-mesh sieve (see Article 214).

210. Test of time of setting. It is important that a cement should not set too quickly or too slowly. A test for time of setting, known as *Gillmore's test,* has been standardized in the United States, and consists in applying to a small cement pat given weights supported by points of specified area (Fig. 178). The cement pat is made by mixing a portion of neat cement with the proper amount of water, mounting this on a piece of glass, and smoothing it until the middle is half an inch thick and the edges are smooth and tapering. The pat is then kept under a damp cloth to prevent injury by sudden changes in temperature, or too high temperature, of the surrounding air. When this pat will hold without appreciable indentation a quarter-pound weight supported by a wire $\frac{1}{12}$ in. in diameter, it is said to have acquired its *initial set.* It is said to have acquired its *final set* when a one-pound weight supported by a wire $\frac{1}{24}$ in. in diameter will not appreciably indent the surface.

When a pat prepared as indicated above checks or warps, it indicates that the cement in setting changes volume too rapidly. For many pieces of work a slow-setting cement cannot be used; but a cement which sets too quickly is likely to contain too much free lime, and should be very carefully tested before being used. In general, the time of final setting for natural cement should not be less than thirty minutes nor more than three hours.

The table given on page 300 shows the time of setting of different brands of cement.* The student is also referred to the standard specifications for cement given in Article 214.

* *Watertown Arsenal Report*, 1901.

TIME OF SETTING OF CEMENTS

Brand of Cement	Water	Time of Setting					
		Gillmore's Method			German Method		
		Initial	Final	Interval	Initial	Final	Interval
	Per cent	*Hr. Min.*	*Hr. Min.*	*Hr. Min.*	*Hr. Min.*	*Hr. Min.*	*Hr. Min.*
Alpha	20	2 20	5 00	2 40	0 35	4 25	3 50
	25	3 20	7 30	4 10	2 50	6 35	3 45
	30	5 40	...	...	4 40	8 40	4 00
Atlas	20	4 05	7 10	3 05	2 45	6 10	3 25
	25	5 10	8 05	2 55	3 35	7 05	3 30
	30	7 00	...	...	5 30	...	...
Star, with plaster	20	2 10	4 25	2 15	0 50	3 00	2 10
	25	4 35	6 00	1 25	3 00	5 30	2 30
	30	5 45	...	...	5 10	7 15	2 05
Star, without plaster	20	0 05	0 15	0 10	0 05	0 10	0 05
	25	0 35	4 55	4 20	0 10	3 30	3 20
	30	5 10	8 35	3 25	3 15	6 50	3 35
Whitehall	20	1 49	5 19	3 30	1 28	4 44	3 16
	25	4 15	6 05	1 50	3 25	5 40	2 15
	30	4 59	7 10	2 20	4 33	6 53	2 20
Josson	20	0 30	4 35	4 05	0 05	3 40	3 35
	25	4 10	6 40	2 30	3 10	6 10	3 00
	30	5 35	8 05	2 30	4 50	7 20	2 30
Storm King	25	4 02	6 57	2 55	1 42	5 37	3 55
	28	5 30	...	...	4 20	7 05	2 45
	30	5 47	...	...	4 27	...	...
Alsen	25	0 25	1 15	0 50	0 10	0 35	0 25
	30	0 30	1 15	0 45	0 20	0 45	0 25
	35	2 30	4 00	1 30	0 30	1 50	1 20
Silica	20	0 20	2 52	2 32	0 13	0 37	0 24
	25	0 29	4 59	4 30	0 22	1 49	1 27
Cathedral	22	4 52	6 17	1 25	1 12	5 32	4 20
	25	4 45	6 55	2 10	2 40	6 10	3 30
	28	5 05	7 29	2 24	3 33	6 45	3 12
Akron Star	30	2 25	6 30	4 05	0 45	4 55	4 10
	35	4 05	7 10	3 05	2 45	6 35	3 50
	40	6 55	...	...	6 15	...	...
Austin	30	0 47	2 51	2 04	0 16	2 08	1 52
	35	1 03	3 18	2 15	0 43	2 28	1 45
	40	1 23	4 48	3 25	1 08	3 58	2 50
Hoffman	30	2 15	3 25	1 10	1 25	2 55	1 30
	35	2 55	5 40	2 45	2 20	4 10	1 50
	40	3 43	...		2 48	...	...
Norton	30	0 37	2 12	1 35	0 25	1 00	0 35
	35	0 49	3 14	2 25	0 34	1 54	1 20
	40	1 02	5 17	4 15	0 40	3 37	2 57
Obelisk	30	1 40	4 05	2 25	0 25	3 20	2 55
	35	2 47	5 02	2 15	1 49	4 12	2 23
	40	3 15	5 20	2 05	2 50	4 15	1 25
Potomac	30	0 45	3 40	2 55	0 25	1 55	1 30
	35	1 05	4 43	3 38	0 43	2 58	2 15
	40	1 15	5 30	4 15	1 07	4 25	3 18
Newark and Rosendale	35	0 37	1 17	0 40	0 32	1 07	0 35
	40	0 47	3 44	2 57	0 40	2 19	1 39
	45	1 08	4 18	3 10	0 48	3 33	2 45
Mankato	40	2 40	5 15	2 35	1 20	3 55	2 35
	45	2 59	...	...	2 29	4 19	1 50
	50	3 24	...	...	2 50	5 09	2 19

211. Test of tensile strength. The tensile strength of a cement is made by testing briquettes of neat cement or cement mortar in tension. The briquettes are made in standard molds (Fig. 180),

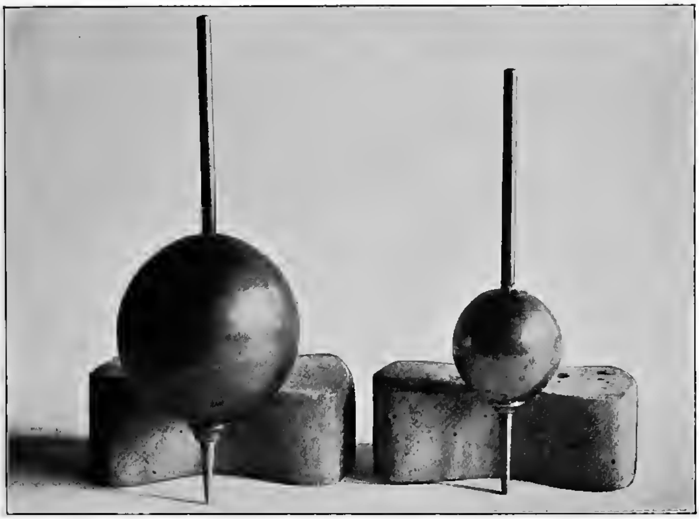


FIG. 178. — Weights for Testing Briquettes

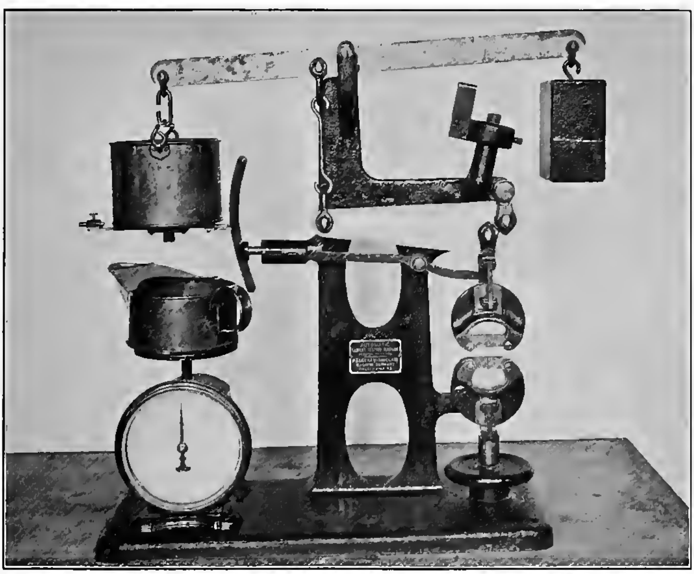


FIG. 179. — Cement Testing Machine

which provide for a cross section of one square inch at the middle, with thicker ends for insertion in the jaws of the testing machine. This test requires considerable expertness to get satisfactory results, for the proper mixing and tamping iuto the molds can only be satisfactorily done by one of considerable experience. After molding, the briquettes are kept under a damp cloth for about twenty-four hours and then under water until tested.

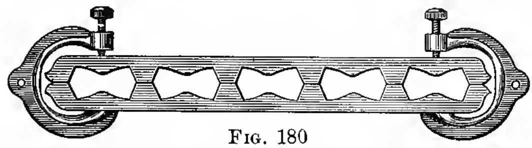


Fig. 180

Many machines are now made for testing the tensile strength of cement, most of them being light enough to be portable. A new automatic machine, manufactured by the Olsen Testing Machine Company of Philadelphia, is shown in Fig. 179. The machine is operated by first placing the briquette in position and balancing the beam at the top. The load is then applied by allowing the shot to run from the pan on the right end of the beam. The spring balance gives the exact weight of the shot and, consequently, the tensile stress on the briquette at any time during the test. After the briquette is broken the tensile strength in pounds per square inch is recorded on the dial.

212. Speed of application of load. It has been found that the rapidity with which the load is applied has considerable effect upon the results obtained in making tension tests of cement. The following table clearly shows this effect.*

EFFECT OF SPEED OF APPLICATION OF LOAD ON TENSILE STRENGTH OF CEMENT

Number of Briquettes	Speed in Pounds-Seconds		Average Results	Number of Briquettes	Speed in Pounds-Seconds		Average Results
	lb.	sec.			lb.	sec.	
129	100	1	560.75	90	100	30	417.27
129	100	15	506.43	90	100	60	403.00
145	100	15	452.2	40	100	60	416.75
145	100	30	430.96	40	100	120	400.00

* *Proc. Inst. Civ. Eng.*, 1883.

TENSILE AND COMPRESSIVE TESTS OF CEMENT

Brand of Cement	Tensile Test			Compression Test				
	Age		lb./in.²	Sectional Area in.²	Age		Total lb.	lb./in.²
	Air (days)	Water (days)			Months	Days		
Atlas	1	6	1066	2.80	1	20	22,050	7,875
	1	6	1012	2.83	1	20	33,700	11,908
	1	6	957	2.89	1	20	33,600	11,626
	1	6	775	2.78	1	20	31,020	11,158
	1	6	759	2.35	1	19	28,100	11,957
	1	6	738	2.90	1	8	25,800	8,896
	1	6	698	2.82	1	8	25,900	9,184
	1	6	654	2.46	1	8	19,500	7,927
	1	6	615	2.92	1	8	26,100	8,938
Storm King .	1	6	174	2.27	1	15	8,300	3,656
	1	6	331	2.56	1	15	10,620	4,148
	1	6	354	2.37	1	15	8,100	3,418
	1	6	189	2.24	1	14	8,640	3,857
	7	. . .	372	2.53	1	14	7,560	2,988
	7	. .	443	2.16	1	14	7,020	3,250
	7	.	543	2.26	1	14	7,540	3,336
Alsen	1	6	714	2.26		24	12,500	5,531
	1	6	670	2.36		24	13,300	5,635
	1	6	755	2.91	. . .	23	16,500	5,670
	7	. . .	446	2.53		23	13,270	5,245
	7	. .	391	2.58	. . .	21	11,300	4,380
Dyckerhoff .	7	. . .	471	2.54	1	8	13,900	5,472
	7		515	2.65	1	8	16,300	6,151
	7	. .	226	2.73	1	7	14,400	5,275
	7	.	279	2.56	1	7	14,300	5,586
Steel . . .	1	6	298	2.22		13	3,840	1,730
	1	6	332	2.67	. .	13	5,300	1,985
	7	. . .	99	2.37	.	13	4,720	1,991
	7		149	2.48	. .	13	4,520	1,822
	7	. .	139	2.20	. .	13	4,100	1,864
Bonneville .	1	6	196	2.39	. .	15	4,100	1,715
Improved .	1	6	158	2.10	.	15	3,180	1,514
	1	6	42	2.22	.	15	3,120	1,405
	1	6	81	2.22	. .	15	3,020	1,360
	7		201	2.14		14	2,710	1,266
	7	.	73	2.24		14	2,710	1,209
	7	. . .	196	2.11		14	2,400	1,137
	7	. .	99	2.03		14	2,520	1,241
Hoffman .	1	6	156	2.48	.	21	5,060	2,040
	1	6	133	2.41		21	4,920	2,041
	7	. .	60	1.94	. .	20	3,030	1,562
	7	. .	53	2.22	. . .	20	3,890	1,752
	7	. .	243	2.56	.	20	4,660	1,820
	7	. .	225	2.35		20	4,550	1,936
	7	. .	275	2.28	. .	20	4,420	1,939

213. Compression tests. Compression tests of cement are made in Europe, but not generally by engineers in the United States, as the tensile test is thought quite as valuable as the compression test in giving results indicative of the strength of the cement. Compression tests are made upon the ends of the specimen broken in tension, or upon specially prepared cement cubes. The use of the broken ends of the briquette insures the same material for the compression test as was used in the tension test. The table on page 302 gives the compressive strength of several brands of cement.* The tests were made by compressing halves of briquettes broken in tension, and both the tensile and compressive strengths are given.

214. Standard specifications for cement. The following is a copy of the standard specifications for cement adopted by the American Society for Testing Materials.

NATURAL CEMENT

This term shall be applied to the finely pulverized product resulting from the calcination of an argillaceous limestone at a temperature only sufficient to drive off the carbonic acid gas.

Fineness. It shall leave by weight a residue of not more than 10 per cent on the No. 100 sieve, and not more than 30 per cent on the No. 200 sieve.

Time of setting. It shall develop initial set in not less than ten minutes, and hard set in not less than thirty minutes nor more than three hours.

Tensile strength. The minimum requirements for tensile strength for briquettes 1 in. square in cross section shall be as follows, and shall show no retrogression in strength within the periods specified.

Neat Cement

Age	Strength
24 hours in moist air	75 lb.
7 days (1 day in moist air, 6 days in water)	150 "
28 days (1 " " " 27 " ") .	250 "

One Part Cement, Three Parts Standard Sand

7 days (1 day in moist air, 6 days in water) . .	50 lb.
28 days (1 " " " 27 " ")	125 "

* *Watertown Arsenal Report*, 1901.

Constancy of volume. Pats of neat cement about 3 in. in diameter, $\frac{1}{2}$ in. thick at the center, tapering to a thin edge, shall be kept in moist air for a period of twenty-four hours.

(*a*) A pat is then kept in air at normal temperature.

(*b*) Another is kept in water maintained as near 70° F. as practicable.

These pats are observed at intervals for at least twenty-eight days, and, to satisfactorily pass the tests, should remain firm and hard and show no signs of distortion, checking, cracking, or disintegrating.

PORTLAND CEMENT

This term is applied to the finely pulverized product resulting from the calcination to incipient fusion of an intimate mixture of properly proportioned argillaceous and calcareous materials, and to which no addition greater than 3 per cent has been made subsequent to calcination.

Specific gravity. The specific gravity of the cement, thoroughly dried at 100° C., shall be not less than 3.10.

Fineness. It shall leave by weight a residue of not more than 8 per cent on the No. 100 sieve, and not more than 25 per cent on the No. 200 sieve.

Time of setting. It shall develop initial set in not less than thirty minutes, and hard set in not less than one hour nor more than ten hours.

Tensile strength. The minimum requirements for tensile strength for briquettes 1 in. square in section shall be as follows, and shall show no retrogression in strength within the periods specified.

Neat Cement

Age	Strength
24 hours in moist air	175 lb.
7 days (1 day in moist air, 6 days in water)	500 "
28 days (1 " " " 27 " ")	600 "

One Part Cement, Three Parts Standard Sand

7 days (1 day in moist air, 6 days in water)	200 lb.
28 days (1 " " " 27 " ") .	275 "

Constancy of volume. Pats of neat cement about 3 in. in diameter, $\frac{1}{2}$ in. thick at the center, and tapering to a thin edge, shall be kept in moist air for a period of twenty-four hours.

(*a*) A pat is then kept in air at normal temperature and observed at intervals for at least twenty-eight days.

(*b*) Another pat is kept in water maintained as near 70° F. as practicable, and observed at intervals for at least twenty-eight days.

(*c*) A third pat is exposed in any convenient way in an atmosphere of steam, above boiling water, in a loosely closed vessel for five hours.

These pats, to satisfactorily pass the requirements, shall remain firm and hard and show no signs of distortion, checking, cracking, or disintegrating.

Sulphuric acid and magnesia. The cement shall not contain more than 1.75 per cent of anhydrous sulphuric acid (SO_3), nor more than 4 per cent of magnesia (MgO).

215. Concrete. When cement mortar is mixed with certain percentages of broken stone, gravel, or cinders, the mixture is called **concrete.** The amount and kind of stone or other material to be used depends upon the use to be made of the finished product. Concrete is rapidly coming into favor as a building material, and is replacing brick and stone in many classes of structures. If properly made it is a much better building material than either of the latter, and has an additional advantage in the fact that it can be handled by unskilled labor and may be readily molded into any desired form. In view of these facts, a study of its properties is of the greatest importance.

216. Mixing of concrete. In making concrete, the sand and cement are first thoroughly mixed and gauged with the right amount of water. The stone, having previously been moistened, is then added, and the whole is thoroughly mixed until each piece of stone is coated with the cement mortar. These two operations are often combined into one. The amount of water to be used in making the mortar depends upon the character of the concrete desired. A medium concrete may be obtained by adding enough water so that moisture comes to the surface when the mortar is struck with a shovel.

After mixing, the concrete is tamped, or rammed, into position. This tamping should be thoroughly done, since in no other way can as dense a mixture be obtained. It is desirable that all the voids (spaces between the broken stone) should be filled as compactly as possible with mortar.

217. Tests of concrete. Concrete is usually tested in compression, and for this purpose 6-inch cubes * are made, composed of cement, sand, and broken stone in the proportions of 1 : 2 : 4 or 1 : 3 : 6. In some cases the proportion to be used in the particular work concerned is also used in making the test cubes. These cubes are made in molds and allowed to set in air, or part of the time in air and the

* Cylinders or larger cubes are also sometimes used.

remainder in water, until tested. The kind of cement as well as its physical properties must be known; also the kind of sand and stone and the degree of fineness of each.

When ready for testing, the concrete cubes are placed in the testing machine, bedded with plaster of Paris or thick paper, and tested in compression. The load at first crack and the maximum load are noted.

The table on the opposite page is a report of a series of tests made at the Watertown Arsenal on Akron Star cement concrete in compression.* It will be noticed that the ultimate strength varied from 600 lb./in.^{2} to 2700 lb./in.^{2}

The table on page 308 is taken from the same volume as the preceding, and summarizes the results of tests on concrete made from different kinds of cement. Various kinds of broken stone were used, including broken brick, and the ultimate strength ranged from 600 lb./in.^{2} to 3800 lb./in.^{2} In making comparisons from the table as to strength several things must be noted, namely, the kind and strength of the cement, the proportions and character of the sand and gravel, the treatment after making, and the age when tested; in other words, a complete history of the materials and their treatment should be known. In the following table the cubes tested were set in air, in a dry, cool place.

The location and character of the structure will often determine the kind of materials to be used in making the concrete. Thus, on account of convenience, pebbles are sometimes used with the sand in which they are found. This reduces the cost of the concrete, but usually impairs its strength, as the proportions of sand and stone as they occur in nature are not likely to be such as to be suitable for concrete. Theoretically, to get the best results the proportions should be such that the cement fills the spaces between the grains of sand, and the mortar fills the spaces between the pieces of stone.

In any particular case the cost of material, strength of the concrete, and service required of the structure must determine what proportions shall be used.

Problem 300. A concrete cube 12 in. high when tested in compression sustained a load of 324,000 lb. at first crack, and 445,200 lb. at failure. Find the intensity of the compressive stress in lb./in.^{2} at first crack and at failure.

* *Watertown Arsenal Report*, 1901.

EFFECT OF VARYING THE RELATIVE PROPORTIONS OF THE INGREDIENTS ON THE COMPRESSIVE STRENGTH OF CONCRETE CUBES

Composition				Age	Dimensions			Sectional Area	Weight Total		Weight per Cubic Foot	Load at First Crack	Ultimate Strength	
Cement	Sand	Gravel	Stone		Height	Compressed Surface							Total	
					in.	in.	in.	sq. in.	lb.	oz.	lb.	lb.	lb.	lb./in.^{2}
1	1½		4	30 days	12.12	12.34	12.08	149.07				215,000	217,200	1457
1	1½		4	"	12.17	12.28	12.10	148.59				216,000	219,400	1477
1	1½		4		12.10	12.42	12.03	149.41				191,000	210,900	1412
1	3		7½	"	12.05	12.30	12.17	149.69				147,000	148,200	990
1	3		7½	"	12.13	12.20	12.04	146.89				153,000	160,600	1093
1	3		7½	"	12.12	11.90	12.08	143.75				131,000	142,300	990
1	2	3	4	"	12.20	12.24	12.12	148.35				174,000	185,100	1248
1	2	3	4	"	12.12	12.20	12.00	146.40				161,000	166,200	1135
1	2	3	4	"	12.13	12.17	12.00	146.04				120,000	132,300	906
1	2	7			12.00	12.18	12.10	147.38				110,000	115,500	784
1	2	7			12.00	12.25	12.12	148.47				98,000	101,300	682
1	2	7			11.96	12.20	12.11	147.74				103,000	114,000	772
1	2¼	8		"	11.96	12.22	12.11	147.98				79,700	89,900	608
1	2¼	8		"	12.15	12.03	12.45	149.77				104,200	112,000	748
1	2¼	8		"	12.12	12.25	12.00	147.00				117,000	126,500	861
1	1½		4	7 months	12.15	12.06	12.04	145.20	150	0	146.92	219,000	227,500	1570
1	1½		4	"	12.16	12.02	12.08	145.20	154	8	151.22	316,000	392,000	2700
1	1½		4	"	12.23	12.06	12.10	145.93	154	4	149.35	309,000	346,000	2370
1	3		7½	"	12.11	12.13	12.12	147.02	151	0	146.56	238,000	298,000	2030
1	3		7½	"	12.14	12.06	12.10	145.93	153	0	149.23	264,000	293,000	2010
1	3		7½	"	12.11	12.11	12.02	145.56	151	12	148.77	274,000	279,500	1920
1	2	3	4	"	11.92	12.12	12.14	147.14	154	4	151.98	302,000	332,600	2260
1	2	3	4	"	12.08	12.14	12.15	147.50	156	4	151.53	335,000	346,000	2350
1	2	3	4	"	12.15	12.02	12.12	145.68	154	4	150.59	281,000	281,000	1930
1	2	7		"	11.98	12.09	12.10	146.29	151	8	149.37	195,700	217,500	1490
1	2	7		"	11.98	12.12	12.13	147.02	147	4	144.47	164,500	186,600	1270
1	2	7		"	11.92	12.12	12.12	146.89	154	12	152.72	273,000	314,500	2140

EFFECT OF DIFFERENT BRANDS OF CEMENT ON THE COMPRESSIVE STRENGTH OF CONCRETE

Brand of Cement	Composition			Age		Weight			Dimension			Sectional Area	Load at First Crack	Ultimate Strength	
	Cement	Sand	Broken Stone	Years	Months	Total		Per Cubic Foot	Height	Compressed surface		in.^{2}	lb.	Total	lb./in.^{2}
						lb.	oz.	lb.	in.	in.	in.			lb.	
Norton's Rosendale .	1	2	4 – 2½″ trap .	1	2	148	0	148.5	11.89	12.06	12.01	144.84	99,000	132,400	914
				1	2	143	0	144.8	11.83	12.02	12.00	144.24	71,000	93,400	648
				1	2	142	8	142.3	11.95	12.04	12.03	144.84	72,000	84,700	585
				1	2	147	8	147.6	11.94	12.03	12.02	144.60	86,760	117,700	814
				1	2	144	0	143.9	11.98	12.01	12.02	144.36	72,400	88,600	614
Alpha Portland .	1	2	4 – 1½″ to 3″ pebbles	1	2	150	0	148.8	12 02	12.04	12.04	144.96	405,888	492,864	3400
				1	2	149	8	148.5	12.01	12.03	12.04	144.84	468,000	529,000	3652
				1	2	148	0	146.5	12.08	12.01	12.03	144.48	471,000	525,000	3634
				1	2	149	0	147.9	12.01	12.03	12.05	144.96	479,000	532,000	3670
				1	2	148	8	146.9	12.06	12.05	12.02	144.84	435,000	493,000	3403
Alpha Portland .	1	2	4 – 1½″ to 2½″ broken brick	1	2	131	0	130.6	11.93	12.02	12.02	144.48	421,000	486,000	3364
				1	2	132	0	131.8	11.92	12.07	12.03	145.20	454,000	488,500	3364
				1	2	130	8	129.8	11.95	12.07	12.04	145.32	368,000	436,000	3000
				1	2	130	8	130.3	11.97	12.03	12.02	144.60	389,000	454,000	3140
				1	2	131	0	130.6	11.98	12.04	12.02	144.72	434,160	501,000	3462
Alpha Portland .	1	3	6 – 1½″ to 2½″ broken brick	1	1	128	8	128.1	12.00	12.03	12.01	144.48	370,000	419,000	2900
				1	1	128	8	127.5	12.05	12.04	12.00	144.48	389,000	435,000	3011
				1	1	128	8	127.0	12.07	12.04	12.03	144.84	336,000	402,300	2778
				1	1	127	0	126.0	11.98	12.08	12.04	145.44	331,000	361,000	2482
				1	1	126	0	125.9	11.92	12.03	12.06	145.08	312,000	364,000	2509
				1	1	125	0	124.8	12.00	12.00	12.02	144.24	270,000	293,500	2035
				1	1	126	8	125.9	12.01	12.02	12.03	144.60	318,120	355,000	2455
Alpha Portland .	1	4	0	1	1	120	0	120.3	11.88	12.03	12.06	145.08	173,000	173,000	1192
				1	1	120	0	120.4	11.87	12.04	12.05	145.08	241,000	241,000	1661
				1	1	120	0	120.8	11.88	12.03	12.01	144.48	202,500	202,500	1401
				1	1	120	8	119.7	11.98	12.04	12.06	145.20	222,000	222,000	1529

218. Modulus of elasticity of concrete. Concrete is so imperfectly elastic that the modulus of elasticity varies with the stress. It also changes with the age of the material and with the change in proportions of cement, sand, and stone.

The variation in the modulus of elasticity with the stress makes it difficult to make theoretical computations in which the modulus of elasticity is involved, as, for instance, in such problems as arise in connection with reënforced concrete beams, etc.*

MODULUS OF ELASTICITY OF CONCRETE IN COMPRESSION

Composition			Age		Modulus of Elasticity in lb./in.² between Loads in lb./in.² of			Compressive Strength
Cement	Sand	Broken Stone	Months	Days	100 and 600	100 and 1000	1000 and 2000	lb./in.²
1	2	4	. .	7	1,000,000	. . . .	. . . .	962
1	2	4	1	. . .	2,083,000	1,875,000	1,190,000	2590
1	2	4	3	. . .	4,167,000	3,750,000	2,778,000	3226
1	2	4	6	. . .	3,125,000	3,214,000	3,571,000	4847
1	3	6	1	. .	2,083,000	2,143,000	. . . .	2364
1	3	6	3	. .	3,571,000	2,812,000	2,000,000	2880
1	3	6	6	. .	4,167,000	4,091,000	1,724,000	2627
1	6	12	1	.	1,667,000	1,607,000	. . .	1452
1	6	12	3	. . .	1,786,000	1,800,000	. . .	1677
1	6	12	6	. . .	1,923,000	1,667,000	. .	2055
1	0	2	. . .	7	2,083,000	1,607,000	1,087,000	2638
1	0	2	1	. . .	3,571,000	3,214,000	2,778,000	3600
1	0	2	3	. . .	2,778,000	2,500,000	1,724,000	3800
1	0	2	6	.	3,571,000	3,461,000	2,381,000	4978
1	2	4	. .	7	2,500,000	2,143,000	1,351,000	2400
1	2	4	1	. . .	. . . .	. . . .	. . . .	2933
1	2	4	3	. . .	3,571,000	3,214,000	2,381,000	3157
1	2	4	6	. .	4,167,000	3,750,000	1,852,000	3309
1	3	6	. . .	7	2,273,000	1,607,000	. . . .	1386
1	3	6	1	. . .	2,273,000	1,875,000	1,219,000	2431
1	3	6	3	. .	2,778,000	2,812,000	2,083,000	2651
1	3	6	6	. .	3,125,000	3,000,000	1,852,000	3207
1	6	12	. . .	7	. . .	. . . .	. .	754
1	6	12	1	. . .	961,000	. . . .	. . .	1088
1	6	12	3	.	2,083,000	1,667,000	. . .	1276
1	6	12	6		1,786,000	1,364,000	.	1097
1	0	2	. .	10	2,500,000	2,368,000	1,515,000	3279
1	0	2	1	.	2,273,000	1,956,000	1,429,000	3420
1	0	2	3	. .	2,273,000	2,250,000	1,515,000	3144
1	0	2	6	. .	3,571,000	3,214,000	2,273,000	5360

* For the method of computing the modulus of elasticity for materials which do not conform to Hooke's law see Article 65.

The strain diagram of concrete in compression, shown in Fig. 181, illustrates the fact that there is no well-defined elastic limit, and that the modulus of elasticity changes as the load increases.

The table on page 309 also illustrates the variation in the modulus of elasticity of concrete in compression.* In the first ten tests the cement used in making the test specimens was *Alpha Portland,* in the next sixteen it was *Germania Portland,* and in the remaining ones *Alsen Portland.*

Problem 301. From the strain diagram of concrete in compression shown in Fig. 181, compute the modulus of elasticity at 1800 lb./in.^{2} and at 2400 lb./in.^{2} The height of the block tested was 10 in.

Problem 302. A concrete beam 6 in. × 6 in. in cross section, and with a 68-in. span, is supported at both ends and loaded in the middle. The load at failure is 1008 lb. Find the maximum fiber stress.

COMPRESSIVE STRENGTH AND MODULUS OF ELASTICITY OF CINDER CONCRETE CUBES

Composition				Age	Compressive Strength lb./in.^{2}	Modulus of Elasticity lb./in.^{2}	
Cement	Sand	Cinders	Water	Days		Between Loads per Square Inch of 500 and 1000 lb.	At Highest Stress Observed
1	2	4	$1\frac{1}{3}$	38	1950	1,786,000	1,136,000
1	2	4	$1\frac{1}{3}$	38	2050	1,923,000	1,136,000
1	2	4	$1\frac{1}{3}$	224	2600	1,471,000	1,087,000
1	2	4	$1\frac{1}{3}$	224	2500	1,563,000	463,000
1	$2\frac{1}{2}$	5	$1\frac{2}{3}$	38	1400	1,250,000	. . . .
1	$2\frac{1}{2}$	5	$1\frac{2}{3}$	38	1400	893,000	. . . .
1	$2\frac{1}{2}$	5	$1\frac{2}{3}$	224	1980	1,136,000	893,000
1	$2\frac{1}{2}$	5	$1\frac{2}{3}$	224	2020	1,250,000	694,000
1	3	6	2	34	1200	781,000	. . . .
1	3	6	2	34	1330	1,000,000	. . . .
1	3	6	2	220	1730	1,000,000	694,000
1	3	6	2	220	1560	735,000	463,000

219. Cinder concrete. The preceding table summarizes the results of a series of tests made on cinder concrete cubes at the Watertown

* *Watertown Arsenal Report,* 1899.

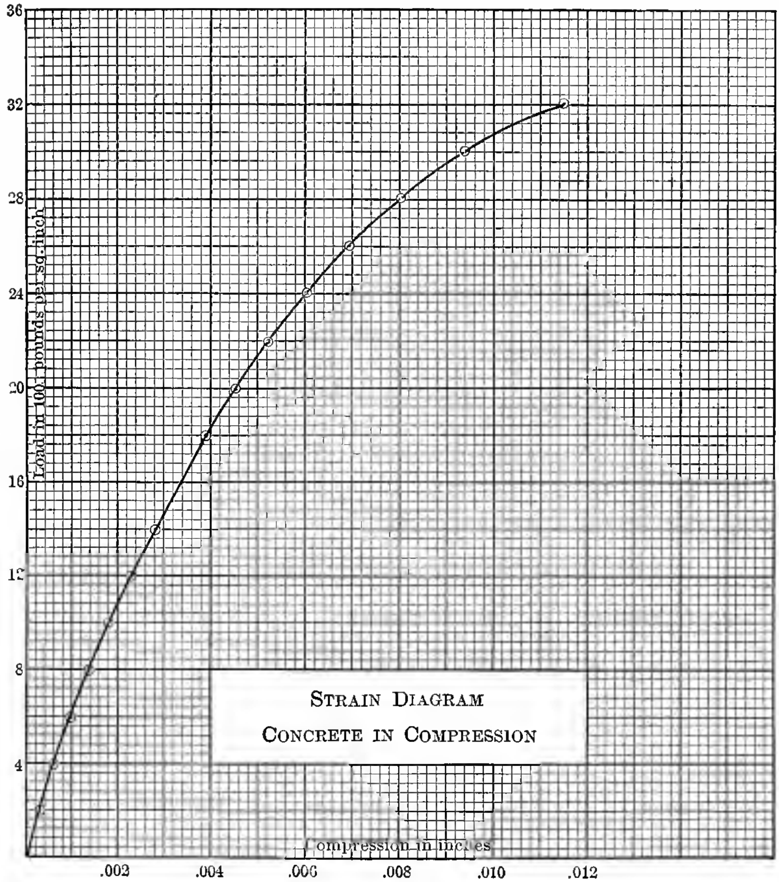


FIG. 181

Arsenal.* The table shows the variation of the modulus of elasticity for different stresses. Lehigh Portland cement was used and the cubes were set in air.

220. Concrete building blocks. During the past few years great progress has been made in the manufacture and use of concrete building blocks. In comparison with stone these have the advantage of cheapness, ease of manipulation, and beauty of the finished product. A type of concrete building block is shown in Fig. 182, and illustrates the general characteristics of such blocks.

* *Watertown Arsenal Report*, 1903.

Few tests have been made on concrete blocks, and but little is known as to their durability. The following table is a report of a series of tests made at the University of Michigan.* The blocks were first tested in flexure, and then an uninjured portion of the block was tested in compression. Blocks 3, 4, 5, and 6 were from the same mixture, and were composed of one part cement, two parts sand, and three parts broken stone. They were all tested after four months.

TESTS OF CONCRETE BUILDING BLOCKS

Number of Block	Distance between Supports in.	Strength in Flexure lb.	Strength in Compression lb./in.²	Strength in Tension lb./in.²
1	13	5450	604	. . . . . .
2	24	3000	1060	. . . .
3	21	4100	705	121
4	21	4000	1500	300
5	21	2900	940	237
6	21	3600	1320	235

Problem 303. A concrete building block 24 in. in length and having an effective cross section of 8 in. × 10 in. minus 4 in. × 10 in. is tested by being supported at both ends and loaded in the middle. The load at failure is found to be 5000 lb. Find the maximum fiber stress, the height of the block being 10 in.

221. Effect of temperature on the strength of concrete. Concrete put in place in cold weather increases in strength very slowly, making it necessary to keep the forms on for a much longer time than is required when the temperature is 70° or over. The failure of many engineers to recognize this fact has been responsible for the early removal of forms in cold weather, and in many such cases for a total or partial collapse of the structure. An investigation recently carried out at the Worcester Polytechnic Institute † shows the rate of increase of the strength of concrete for temperatures ranging from 32° F. to 74° F. as summarized in the following table.

Age in Days	Tensile Strength, lb./in.²			
	32° F.	36° F.	47° F.	74° F.
1	608	679	897	1059
2	1027	1232	1435	1746
3	1127	1312	1443	1491
4	1274	1422	1587	1598
5	1591	1662	1790	1359 ‡
6	1617	1689	. . . . . .	1524 ‡

* *Concrete*, February, 1905. † *Eng. News*, Vol. LXII, p. 183. ‡ Probably dried out.

FIG. 182. — Concrete Building Blocks

CHAPTER XIV

REËNFORCED CONCRETE

222. Object of reënforcement. The fact that concrete is much stronger in compression than in tension has led to attempts to increase its tensile strength by imbedding steel or iron rods in the material. This metal reënforcement is so designed as to carry most of the tensile stress, and thus plays the same part in a concrete structure as the tension members play in a truss.

It has been found by experiment that reënforced concrete beams may be stressed in flexure far beyond the elastic limit* of ordinary concrete, and even beyond the stress which would rupture the same beam, if not reënforced, without appreciable injury to the material. M. Considère, one of the leading French authorities on the subject, reports a test of this kind, in which he found that concrete taken from the tensile side of a reënforced concrete beam tested in flexure was uninjured by the strain. Professor Turneaure, of the University of Wisconsin, has found that minute cracks occur on the tension side of a reënforced concrete beam as soon as the fiber stress reaches the point at which non-reënforced concrete would crack.† Experiments of this kind seem to indicate that the metal reënforcement carries practically all of the tensile stress, as cracks in the concrete must certainly reduce its tensile strength to zero at this point.

223. Corrosion of the metal reënforcement. The maintenance of the increased strength of concrete due to the metal reënforcement depends upon the preservation of the metal. The corrosion of metal imbedded in concrete is thus a matter of the greatest importance in connection with reënforced concrete work. It has been found that metal thus protected does not corrode even though the concrete be

* As indicated in Chapter XIII, concrete shows no well-defined elastic limit, i.e. the material does not conform to Hooke's law. In this case elastic limit means the arbitrary point beyond which the deformations are much more noticeable than formerly.

† *Proc. Amer. Soc. for Testing Materials*, 1905.

subjected to the severest exposure. However, the existence of cracks on the tension side of reënforced beams makes the exposure of the metal rods possible, and thus adds a new danger to the life of the beam; but the small hairlike cracks that occur after the elastic limit of the concrete has been passed probably have no effect in this respect. When they become large enough to expose the reënforcement, the strength of the beam is endangered.

224. Adhesion of the concrete to the reënforcement. When a reënforced concrete beam is subjected to stress, there is always a tendency to shear horizontally along the reënforcement. This is prevented in part by the adhesion between the steel and concrete. Failure sometimes occurs, due to this horizontal shear, especially when the beam is over-reënforced, i.e. when the area of cross section of the reënforcement is large as compared with the total area of cross section of the beam. When plain round or square rods are used, the adhesion between the steel and concrete furnishes the only bond. For commercial purposes, however, various forms of reënforcement are ordinarily used to increase this bond. Four of these commercial types are illustrated in Fig. 183. The Johnson, Thacher, and Ransome bars are provided with projections and indentations to prevent the bar from pulling out of the concrete, while the Kahn bar, by means of the projecting arms that extend upward along the lines of principal stress in the beam, is also designed to act as a truss. Several other commercial types of bar are also in use, but all are provided with projections or indentations of some kind to prevent slipping.

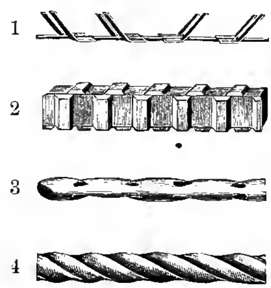


Fig. 183

1, Kahn trussed bar; 2, Johnson corrugated bar; 3, Thacher bulb bar; 4, Ransome twisted bar

Many tests have been made to determine the force necessary to pull the various forms of rods from concrete. The following table gives the results of pulling-out tests made by Professor Edgar Marburg, of the University of Pennsylvania.* The rods in this case were imbedded centrally in 6 in. × 6 in. concrete prisms 12 in. long, and were tested after thirty days. In most cases, except in that of the

* *Proc. Amer. Soc. for Testing Materials*, 1904.

plain rods, failure was due to the breaking of the rods or the cracking of the concrete. On account of the projections on some of the rods these can hardly be called adhesion tests, but should more properly be called pulling-out tests.

As might be expected, the plain rods show the lowest values, since any reduction in cross section of the rod, due to the tensile stress upon it, largely destroys the adhesion of the concrete. Square reënforcing rods, or those that present sharp angles, are likely to cause initial cracks upon the shrinkage of the concrete. To have the strongest bond a rod should be round, with rounded projections.

PULLING-OUT TESTS

Kind of Rod	Total Load lb.	Load per Linear Inch of Rod lb.	Remarks
Johnson	13,660	1138	Elastic limit passed. Concrete cracked.
	12,830	1069	Elastic limit passed. Concrete cracked.
	9,980	832	Concrete cracked.
Plain	6,280	524	Rod pulled out.
	6,190	516	Rod pulled out.
	5,650	471	Rod pulled out.
Thacher	10,420	868	Rod broke.
	8,890	741	Concrete cracked.
	9,970	831	Rod broke.
Ransome	22,690	1891	Concrete cracked.
	16,680	1390	Concrete cracked.
	19,290	1608	Rod pulled out.

225. Area of the metal reënforcement. Since the small hairlike cracks mentioned in Article 222 occur early during the flexure of a reënforced concrete beam, it is evident that in designing little can be allowed for the tensile strength of the concrete. The problem becomes one of opposing the compressive strength of the concrete and the tensile strength of the reënforcement. This means that knowing the safe compressive strength of the concrete and the area of the concrete in compression, sufficient steel must be used to carry safely a tensile load equal to the compressive load on the concrete. Professor Marburg, in the paper referred to in the preceding article,

gives 1600 lb./in.^{2} for the compressive strength of 6-inch cubes thirty days old. A slightly higher value was found for cubes from a different mixture.

From an investigation of the tensile strength of steel reënforcing bars, the writer referred to above obtained the following values.

Type of Rod	Area of Metal in.^{2}	Elastic Limit lb./in.^{2}	Ultimate Strength lb./in.^{2}	Modulus of Elasticity lb./in.^{2}	Percentage of Elongation in 8 in.
Plain . .	.75	40,500	60,600	30,500,000	23.50
Johnson .	.54	65,800	102,300	28,500,000	13.50
Ransome .	.76	58,000	86,500	26,000,000	7.75
Thacher .	.59	31,900	51,300	28,500,000	13.00

With a 1–3–6 concrete a 1.5 per cent reënforcement of steel, having an elastic limit of 33,000 lb./in.^{2}, and a 1.0 per cent reënforcement of steel, having an elastic limit of 55,000 lb./in.^{2}, has been used without developing the full compressive strength of the concrete.* In this case the percentage is figured on the area of concrete above the center of the metal reënforcement. This percentage may also be figured on the area of cross section of the beam.

226. Position of the neutral axis in reënforced concrete beams. In Article 218 it was pointed out that the modulus of elasticity of concrete in compression is not constant. This indicates that in the case of flexure the position of the neutral axis changes with the stress, at first lying near the center, but moving toward the compression side as the load is increased. In a reënforced concrete beam the neutral axis also undergoes a displacement, due to the non-homogeneity of the cross section, since the moduli of elasticity of steel and concrete are not the same. In this case, if the beam is reënforced only on the tension side, and the metal reënforcement is designed to carry all the tensile stress, the neutral axis usually lies nearer the tension side of the beam than the compression side.†

From tests made at Purdue University, Professor Hatt found the ratio of the moduli of elasticity of steel in tension to concrete in

* *Proc. Amer. Soc. for Testing Materials*, 1905.

† See article by S. E. Slocum, entitled "Rational Formulas for the Strength of a Concrete Steel Beam," *Engineering News*, July 30, 1903.

compression, for certain grades of material to be as follows.* The use of this ratio is exemplified in the following article.

Stone concrete	23 days	8.8
Stone concrete	90 days	6.6
Average		7.7
Gravel concrete	28 days	8.0
Gravel concrete	90 days	6.2
Average		7.1

227. Strength of reënforced concrete beams. Concrete is weak in tension and strong in compression, so that when used in the form of a beam, the tensile strength controls the strength of the beam. To correct for this lack of strength in tension, steel rods are imbedded in beams in such a way as to carry the tensile stresses.

A few years ago engineers believed that the tensile strength of the concrete in a beam might be considered in computing the strength of the reënforced concrete beam. At present, however, the steel reënforcement is designed to carry all the load in tension and the concrete all the load in compression; that is, the tensile strength of the steel is balanced against the compressive strength of the concrete. Since concrete is imperfectly elastic, the stress-strain diagram is not quite a straight line in any part of its length. This means that its modulus of elasticity is not constant, but changes with the stress. The results of many tests show that the stress-strain diagram for concrete may be assumed a parabola, so that the compressive stress on any section of the beam varies as the ordinates of a parabola.

On account of the difference in the modulus of elasticity of steel and concrete (30,000,000 lb./in.^{2} and 2,000,000 lb./in.^{2} to 2,500,000 lb./in.^{2}), the position of the neutral axis changes with the load on the beam. In the following analysis the assumptions of the common theory of flexure are supposed to hold, with the exception of the points stated above.

Let l = length of span,
x = distance of the neutral axis from the compression face,
d = effective depth of beam; that is, the distance from top of beam to center of gravity of reënforcement,

* *Jour. Western Soc. Eng.*, June, 1904.

r = ratio of area of steel to that of the effective cross section of the beam,
E_s = modulus of elasticity of steel,
E_c = modulus of elasticity of concrete in compression,
p_s = unit stress in metal reënforcement,
p_c = unit compression stress in the concrete at outer fiber,
e = unit contraction in concrete, and e' unit elongation in the steel,
E_c is measured at stress p_c.

The beam is supposed reënforced on the tension side only, and the rods are imbedded to a sufficient depth to protect the steel. (This

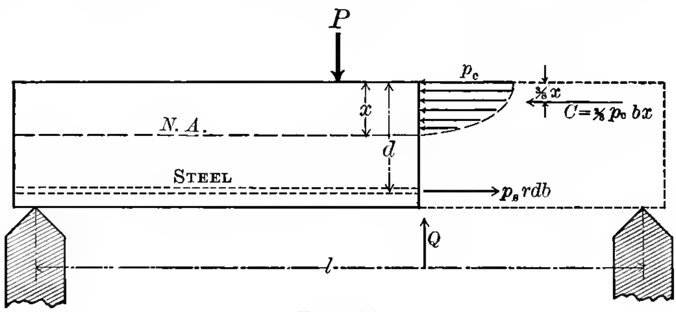


FIG. 184

depth may be as much as $2\frac{1}{2}$ to 3 in. if the best fire protection is desired.) The illustration (Fig. 184) shows the beam supported at the ends and loaded in the middle, but the formulas derived apply to beams having different loadings and supports.

Considering the compressive stress to vary as the ordinates of a parabola, the total compressive stress in the concrete is $C = \frac{2}{3} p_c bx$, and it may be considered as acting $\frac{3}{8} x$ (distance of center of gravity of the parabolic area) from the top of the beam. The total stress in the steel is $p_s rbd$.

The moment of the stress couple may therefore be written

$$M = \frac{2}{3} p_c xb \left(d - \frac{3}{8} x\right), \tag{a}$$

or

$$M = p_s rdb \left(d - \frac{3}{8} x\right); \tag{b}$$

that is, either the compression in the concrete times the distance $(d - \frac{3}{8} x)$, or the tension in the steel times the same distance. This

moment of the stress couple must be equal to the moment of the external forces acting upon either portion of the beam. If the beam is supported at the ends and loaded in the middle, and the middle section is considered, then

$$\frac{2}{3}p_c xb\left(d-\frac{3}{8}x\right)=\frac{Pl}{4}, \qquad (c)$$

or

$$p_s rbd\left(d-\frac{3}{8}x\right)=\frac{Pl}{4}.$$

Equating the summation of horizontal forces (Fig. 184) to zero, we have

$$\tfrac{2}{3}p_c x = p_s rd,$$

or

$$x=\frac{3\,p_s r}{2\,p_c}. \qquad (d)$$

But on the assumption that the cross sections of the beam remain plane during flexure (see Fig. 185),

$$\frac{e}{x}=\frac{e'}{d-x}, \quad \text{or} \quad \frac{e}{e'}=\frac{x}{d-x}.$$

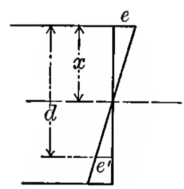


FIG. 185

From definition,

$$E_c=\frac{p_c}{e} \quad \text{and} \quad E_s=\frac{p_s}{e'},$$

so that

$$\frac{e}{e'}=\frac{p_c E_s}{p_s E_c},$$

and therefore

$$\frac{x}{d-x}=\frac{p_c E_s}{p_s E_c}. \qquad (e)$$

Eliminating p_c between (d) and (e), we have

$$\frac{2}{3}x^2=rd\frac{E_s}{E_c}(d-x). \qquad (f)$$

Equation (f) may be used to locate the neutral axis in a beam. The ratio $\frac{E_s}{E_c}$ is taken by different authorities from 12 to 15. Tests made in this country seem to show the lower value as more nearly correct, although 15 is usually used. It has also been found that the relations of d and x may be expressed approximately by

$$x=.52\,d.$$

Substituting this value in equations (*a*) and (*b*), we have

$$M = .277\, p_c bd^2, \tag{g}$$

and

$$M = .80\, p_s rbd^2. \tag{h}$$

Problem 304. A reënforced concrete beam 8 in. × 10 in. in cross section, and 15 ft. long, is reënforced on the tension side by six $\frac{1}{2}$-in. plain steel rounds. The steel has a modulus of elasticity of 30,000,000 lb./in.^{2}, and the center of the reënforcement is placed 2 in. from the bottom of the beam. Assuming that $E_c = 3{,}000{,}000$ lb./in.^{2}, and $p_c = 600$ lb./in.^{2}, find from formulas (*f*) and (*a*) the position of the neutral axis and the moment M.

NOTE. The moment M corresponds to the moment $\frac{pI}{e}$ obtained from the consideration of the flexure of homogeneous beams; that is to say, M is the moment of resistance of the beam (see Article 44).

Problem 305. For a stress $p_c = 2700$ lb./in.^{2} on the outer fiber of concrete in the beam given in Problem 304, find the stress p_s in the steel reënforcement.

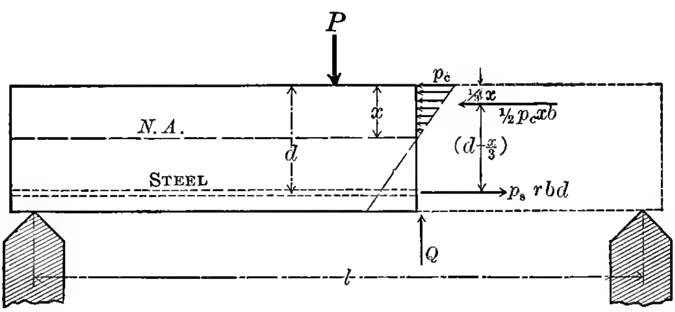


FIG. 186

Problem 306. Using the data of Problem 304, locate the neutral axis, and find the value of the moment of resistance M under the assumption that the stresses in the concrete vary linearly.

228. Linear variation of stress. It is believed by most engineers that it is not necessary to consider that the compressive stress varies as the ordinates of a parabola, but that, for working loads, the linear variation is close enough for practical purposes (see Fig. 186).

Equations (*a*) and (*b*) may then be written

$$M = \frac{1}{2} p_c xb \left(d - \frac{x}{3}\right), \tag{i}$$

$$M = p_s rbd \left(d - \frac{x}{3}\right), \tag{j}$$

and equation (f) becomes

$$\frac{1}{2}x^2 = dr\frac{E_s}{E_c}(d - x).$$

In this case it has been found that

$$x = \tfrac{3}{8}d, \text{ approximately,}$$

so that equations (i) and (j) may be written

$$M = .164\, p_c bd^2, \tag{k}$$

$$M = .875\, p_s rbd^2. \tag{l}$$

229. Bond between steel and concrete. The reënforced concrete beam should be regarded as a girder. The concrete in compression should be regarded as one flange, the steel in tension as the other, while the web is made up of concrete. In order that the steel reënforcement may act effectively, it is necessary that there be sufficient bond between the steel and concrete to carry the horizontal shear occurring along the reënforcement. The stress that this bond must carry is about the same as that carried by the rivets connecting the flange and web in a plate girder.

If y denotes distance along the beam, we know (Article 53) that

$$\frac{dM}{dy} = Q, \tag{m}$$

and so from equations (a) and (m),

$$Q = \frac{dp_s}{dy}F\left(d - \frac{3}{8}x\right),$$

where F is the area of cross section of the reënforcement, or, calling $d - \frac{3}{8}x$, d', this may be written

$$F\frac{dp_s}{dy} = \frac{Q}{d'}.$$

But $F\dfrac{dp_s}{dy}$ is the rate of change of *total stress* in the reënforcing bars as y varies along the beam. For unit length of beam, it measures the stress transmitted by the concrete to the bars, that is, the *bond*.

Let k = number of bars,

and o = surface of one bar per inch of length.

Then ok = surface of steel per inch of length of beam,

and oku = bond, where u is the bond developed per unit area of rod surface of bars.

Then $$\frac{Q}{d'} = oku.$$

For parabolic loading,

$$u = \frac{Q}{(d - \frac{3}{8}x)\,ok}. \tag{n}$$

If $x = .52\,d$,

$$u = \frac{Q}{.8\,(okd)}. \tag{o}$$

For the linear variation,

$$u = \frac{Q}{\left(d - \frac{x}{3}\right)ok}. \tag{p}$$

If $x = \frac{3}{8}\,d$, this becomes

$$u = \frac{Q}{\frac{7}{8}\,(okd)}. \tag{q}$$

These equations give the unit horizontal shearing stress along the reënforcement. From what has been shown previously, this is also the unit vertical shearing stress at the reënforcement.

Turneaure and Maurer * give the following as *working* stresses in concrete beams.

Ultimate Strength, Concrete Compression lb./in.^{2}	Working Stress, Concrete Compression lb./in.^{2}	Elastic Limit, Steel lb./in.^{2}	Safe Stress in Steel lb./in.^{2}	Safe Bond lb./in.^{2}
2000–2200	500–600	32,000	12,000–15,000	50–75 for plain; 100 for deformed

The weight of concrete . . . . 140–150 lb./ft.^{3}
The weight of reënforced concrete . . . . . . 150 lb./ft.^{3}

Problem 307. A concrete beam is 10 × 16 in. in cross section and 20 ft. long. It is reënforced with four $\frac{3}{4}$-in. steel rods with centers 2 in. above the lower face of the beam. The safe compressive strength of the concrete is 600 lb./in.^{2} and the steel used has an elastic limit of 40,000 lb./in.^{2} What single concentrated load will the beam carry at its middle? What tension will be developed in the steel? What shearing stress along the reënforcement?

Problem 308. Find what load, uniformly distributed, the beam in the preceding problem will carry, and find the tension in the steel and bond for this case.

230. Strength of T-beams. The T-beam shown in Fig. 187 is much used in floor systems in reënforced concrete buildings. Here, as in

* *Principles of Reinforced Concrete Construction*, pp. 170–172.

the case of those of rectangular section, the cross sections are assumed to remain plane during bending. We have, then,

$$\frac{e}{e'} = \frac{p_c E_s}{p_s E_c} = \frac{x}{d - x}. \tag{r}$$

The tension of the concrete in the web, and the small amount of compression when the neutral axis falls below the flange, may be

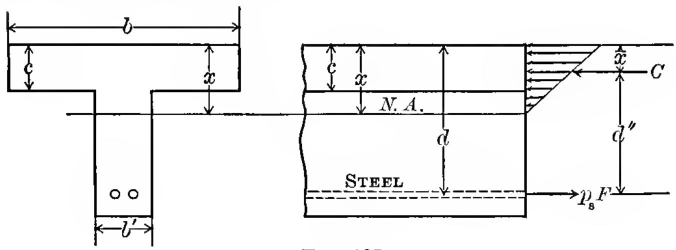


FIG. 187

neglected without serious error. Then the compression in the flange is balanced by the tension in the steel, or $C = p_s F$. The average unit stress in the flange $p_c' = \frac{p_c}{x}\left(x - \frac{c}{2}\right)$ and the total compressive stress $C = \frac{p_c}{x}\left(x - \frac{c}{2}\right)bc$, so that

$$\frac{p_c}{x}\left(x - \frac{c}{2}\right)bc = p_s F. \tag{s}$$

Eliminating $\frac{p_s}{p_c}$ by means of (r), we have for x,

$$x = \frac{2\,nFd + cA}{2\,(A + nF)},$$

where $n = \frac{E_s}{E_c}$ and $A =$ area of flange.

The moment of the internal couple is

$$M = cd'' = p_s F d''.$$

The distance $d'' = d - \bar{x}$, where $\bar{x}$ is the distance of the center of gravity of the trapezoidal area of stress in the flange from the outer fiber. This distance $\bar{x}$ may be expressed as

$$\bar{x} = \frac{c}{3} \cdot \frac{3\,x - 2\,c}{2\,x - c}.$$

The resisting moment of the T-beam may now be written

$$M = C(d - \bar{x}) = \frac{p_c}{x}\left(x - \frac{c}{2}\right)bc(d - \bar{x}), \tag{t}$$

or

$$M = p_s F(d - \bar{x}). \tag{u}$$

If the neutral axis falls within the flange, then d'', the arm of the internal couple, will be greater than $d - \frac{c}{3}$, so that a safe approximation for the resisting moment is obtained by using

$$M = \frac{p_c bc}{2}\left(d - \frac{c}{3}\right), \tag{v}$$

or

$$M = p_s F\left(d - \frac{c}{3}\right). \tag{w}$$

When the neutral axis falls on the lower edge of the flange, these formulas (*v*) and (*w*) are exactly true.

The horizontal shear in the case of T-beams may be obtained as follows. From equation (*u*)

$$\frac{dM}{dy} = \frac{dp_s}{dy} F(d - \bar{x}) = Q,$$

and

$$\frac{Q}{d - \bar{x}} = oku,$$

where oku has the same meaning as in the case of rectangular beams. So that

$$u = \frac{Q}{ok(d - \bar{x})},$$

or, from (*w*),

$$u = \frac{Q}{ok\left(d - \frac{c}{3}\right)}.$$

231. Shear at the neutral axis. If the tension in the concrete is neglected, in the case of rectangular beams, the horizontal shear at the neutral axis must be equal to the horizontal shear along the reënforcement. If u' is the unit horizontal shearing stress in the

concrete at the neutral axis and b is the width of the beam, then $bu' =$ shear per unit length of beam, at the neutral axis, and so

$$bu' = \frac{Q}{d'},$$

and therefore for parabolic variation of stress

$$u' = \frac{Q}{b\,(d - \frac{3}{8}\,x)},$$

and for linear variation of stress

$$u' = \frac{Q}{b\left(d - \frac{x}{3}\right)}.$$

In the case of the T-beams, if the tension in the web is neglected, the horizontal shear where the web joins the flange must be equal to the shear along the steel. Then

$$b'u' = \frac{Q}{d - \bar{x}},$$

and so

$$u' = \frac{Q}{b'\,(d - \bar{x})}.$$

CHAPTER XV

BRICK AND BUILDING STONE

232. Limestone. Limestone is principally a carbonate of lime, made up of seashells that have been deposited from water during past geological times. Its method of formation has much to do with its value as a building material. If it contains no thin layers of clay or shale (sedimentary planes), it is likely to be fairly homogeneous in structure. But if layers of shale, however small, occur, the material is much more quickly weathered. This is especially true if the stone be placed at right angles to the position it occupied in the quarry.

Thin planes of foreign substances are likely to occur in many of our best building stones, as may be seen in the rapid deterioration of seemingly first-class limestone when used as curbing. Such disintegration is caused by a lessening of the adhesion between the particles of stone.

Limestone may be composed of a great percentage of sand cemented together by calcareous matter, in which case it is called *siliceous limestone.* Under such circumstances chemical action may remove the cementing material, thus leaving the stone free to crumble. Marble is almost pure limestone.

Conditions to which a building stone is to be exposed will determine the character of the material to be used in any particular structure. Rapid freezing and thawing is likely to set up internal strains in the material, which may lead to future failure. These strains may be caused by unequal expansion or contraction of the particles of the stone, or by the freezing and thawing of the water in the stone. The formation of ice in the sedimentary planes accounts in a large measure for the rapid deterioration of stone.

Limestone often occurs in very thick layers, as in the case of the oölitic limestone found at Bedford, Indiana, where the layers are often from 25 to 30 ft. thick. In such cases it is a most valuable

building stone, especially for bridge piers and other structures where large masses of stone are needed. This particular limestone, unlike most others, is easily worked, being almost equal to sandstone in this respect.

When limestone is subjected to the atmosphere of a large city, where great quantities of coal are used, it is acted upon by the sulphuric acid in the air. To determine the effect of this action, a small piece of stone, well cleaned, is placed in a 1 per cent solution of sulphuric acid and left for several days. If no earthy matter appears, it may be concluded that the stone will withstand the action of the atmosphere.

233. Sandstone. Sandstone consists very largely of grains of sand (silicon) cemented together. It has been deposited from water, making it homogeneous in structure, and as it occurs in vast beds, it is very suitable for building purposes. The ease with which it may be carved and worked makes it a much more valuable building material than limestone. Various foreign substances, such as iron, manganese, etc., give to the stone a variety both in color and texture. Sandstone absorbs water much more readily than limestone, and were it not for the fact that it occurs in such thick layers, and is therefore almost free from sedimentary planes, this might be a serious objection to its use. The mean weight of sandstone is 140 lb./ft.^{3}; that of limestone is 160 lb./ft.^{3}

234. Compression tests of stone. The most common test for a building stone is that of subjecting it to a direct crushing force in an ordinary testing machine. To prevent local stresses, the specimen, which is generally a well-finished cube, is usually bedded in plaster of Paris, thin pine boards, or thick paper, and the load at first crack and the maximum load are noted. The friction of the bedding against the heads of the machine tends to prevent the spreading of the specimen near these heads and thus adds to the strength of the cube. Great care is necessary in preparing the specimen, in order to get the two bearing faces exactly parallel. The stone fractures along the 30° line approximately, giving the characteristic fracture of two inverted pyramids (Figs. 188 and 189).

From a series of tests made by Buckley on the building stones of Wisconsin,* the average of ten tests on limestone gave an ultimate

* Buckley, *Building Stones of Wisconsin.*

strength of 23,116 lb./in.², a modulus of elasticity ranging from 31,500 lb./in.² to 1,800,000 lb./in.², and a shearing strength ranging from 1735 lb./in.² to 2518 lb./in.² The average of thirty tests on sandstone gave an ultimate strength of 4109 lb./in.², and a modulus of elasticity ranging from 32,000 lb./in.² to 400,000 lb./in.²

From a series of tests on building stone from outside the state of Wisconsin, the same report gives the ultimate strength of limestone as ranging from 3000 lb./in.² to 27,400 lb./in.², and the ultimate strength of sandstone from 2400 lb./in.² to 29,000 lb./in.² This report also gives tables showing the effect of freezing and thawing on the strength of stone, the effect of sulphuric acid on limestone, and the effect of high temperatures on building stone.

The following table shows the results of a series of compressive tests made upon limestone at the Watertown Arsenal.*

HEIGHT in.	SECTIONAL AREA in.²	FIRST CRACK lb.	ULTIMATE STRENGTH lb./in.²
4.06	16.4	361,000	28,950
4.08	16.36	178,000	18,496
3.99	15.88	217,200	13,680
3.99	16.04	219,100	13,660
4.01	15.96	241,000	15,320
4.00	15.96	273,400	17,130

From another series of tests made at the Watertown Arsenal on a different grade of limestone, the average value of the ultimate strength was found to be 7647 lb./in.², and the modulus of elasticity to be 3,200,000 lb./in.² †

This wide range in the strength of building stone is explained by the method of its formation, which makes the character of the stone from one locality often differ entirely from that of a neighboring locality. Average values of the strength of building stone are therefore of little value, and must be used with a large factor of safety.

Problem 309. A granite block was tested in compression, the load at first crack and at maximum being 263,000 lb. and 417,400 lb. respectively. The sectional area was 16.4 in.² Find the intensity of stress at first crack and at maximum.

* *Watertown Arsenal Report*, 1900. † *Watertown Arsenal Report*, 1894.

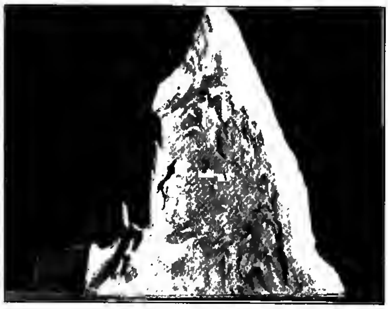


Fig. 188. — Result of Compression Test of Limestone

Fig. 189. — Results of Compression Tests of Sandstone

235. Transverse tests of stone. The use of stone where transverse ress is applied calls for some knowledge of its transverse strength. stone may meet the specifications for crushing and yet fail entirely hen subjected to cross bending, since a beam is in tension on one le and in compression on the other. As stone is much stronger in mpression than in tension, it usually fails in tension under transrse loading.

To test the transverse strength of stone, small beams are pre-red usually 1 in. square by 6 or 8 in. long. These are supported knife-edges resting on the platform of the testing machine, and e load is applied at the center. Buckley reports limestone beams in. × 1 in. × 6 in. to have a modulus of rupture of 2000 lb./in.^{2}, id sandstone beams 1 in. × 1 in. × 4 in. to have a modulus of rupture 1000 lb./in.^{2}

236. Abrasion tests of stone. The most extended series of tests stone in resisting abrasion was made by Bauschinger.* Four-inch ibes under a pressure of 4 lb./in.^{2} were subjected to the abrasive tion of a disk having a radius of 19.5 in. and making 200 revotions per minute, upon which 20 g. of emery were fed every 10 volutions. The loss of volume in cubic inches was as follows.

Granite	.24	dry and	.46	wet
Limestone	1.10	"	1.41	"
Sandstone	.80	"	.64	"
Brick	.38	"	.75	"
Asphalt	.60	"	1.62	"

brasion tests of stone have never been standardized, and comparison results of different tests must be made with a full understanding all the conditions affecting the results.

237. Absorption tests of stone. The absorption test is made to termine the amount of water absorbed by the dry stone. In making e test the specimen is first heated for several hours at a temperare of 212° F., and then placed in water for about thirty hours. The crease in the weight of the specimen divided by its weight when y and multiplied by 100 gives the percentage by weight of moisture sorbed. This percentage for a series of tests varied, for granite, om 1.1 to .3; for limestone, from 3.6 to 1.2; and for sandstone, from 3.8 to 1.6.

* *Communications*, 1884.

238. Brick and brickwork. Brick is generally made by tempering clay with the proper amount of water, and then molding into the desired shape and burning. The tempered clay is used *wet, dry,* or *medium,* depending upon the kind of brick desired, and these are classified as *soft mud* brick, *pressed* brick, or *stiff mud* brick respectively. The position of the brick in the kiln may also determine its classification as *hard* brick, taken from nearest the fire, *medium* brick from the interior of the pile, and *soft* brick from the exterior of the pile.

Paving brick is a vitrified clay brick or block somewhat larger than the ordinary brick.

239. Compression tests of brick. For this test a whole or half brick is tested edgewise or flat in much the same way as in the crushing test for building stone. The faces which are to be in contact with the heads of the testing machine are ground perfectly smooth and parallel, or are bedded, or both. If plaster of Paris is used, it should be placed between sheets of paper to prevent the absorption of water by the brick, as this may affect its strength. In any case, in testing brick or stone in compression it is desirable to use a spherical compression block for one of the heads, so that in case the faces of the test piece are not parallel the bearing will adjust itself to bring the axis of the test piece into coincidence with the axis of the machine. In this case, also, the load at first crack and the maximum load are noted. The form of the fractured specimen is also noted; it is usually that of the double inverted pyramid. An imperfect bedding may cause the specimen to split vertically into thin pieces. Cardboard cushions and soft pine boards are also used in bedding brick for testing.

The relative value of the kinds of bedding, as indicated by tests made at the Watertown Arsenal * on half bricks, may be seen from the following table.

	MEAN STRENGTH
Set in plaster of Paris	5640 lb./in.^{2}
Set in cardboard cushions	4430 "
Set in pine wood	4540 "

The strength of a single brick in compression cannot be taken as a criterion of its strength in an actual structure, since its strength in that case must depend somewhat upon the mortar used. If the mortar is soft and flows (i.e. is squeezed out), the brick may fail in

* *Watertown Arsenal Report,* 1901.

tension, due to the lateral flow of mortar, instead of in compression. From a series of thirty-eight tests made at the Watertown Arsenal * on piers of common brick, it was found that the maximum compressive strength varied from 964 lb./in.^{2} to 2978 lb./in.^{2} The mortar in this case was composed of one part Rosendale cement and two parts sand. The bricks used in these piers developed only one half their compressive strength. The compressive strength of soft brick may go as low as 500 lb./in.^{2}, and that of paving brick as high as 15,000 lb/in.^{2}, when used in piers.

The following table gives the results of tests of the compressive strength of common brick made at the Watertown Arsenal. The compressed surfaces were bedded in plaster of Paris, and the bricks were tested whole.

COMPRESSIVE STRENGTH OF COMMON BRICK

Number of Brick	Dimensions			Sectional Area in.^{2}	Load at First Crack lb.	Ultimate Strength	
	Height in.	Compressed Surface				Total lb.	lb./in.^{2}
		in.	in.				
1	2.50	4.22	8.43	35.57	86,000	186,900	5,250
7	2.48	4.12	8.57	35.31	107,000	257,200	7,280
13	2.33	3.99	8.47	33.80	269,000	309,500	9,150
19	2.27	4.04	8.19	33.09	442,000	609,000	18,400
22	2.30	4.02	8.19	32.92	93,000	446,100	13,550
25	2.43	4.11	8.67	35.63	108,000	234,800	6,590
28	2.32	4.09	8.36	34.19	191,000	361,500	10,570
31	2.55	4.02	8.51	34.21	216,000	312,000	9,120
34	2.41	4.18	8.48	35.45	143,000	181,000	5,110
37	2.38	4.00	8.33	33.32	164,000	282,500	8,480
43	2.46	4.12	8.57	35.31	63,000	224,500	6,360
45	2.41	4.14	8.57	35.48	96,000	242,700	6,840
48	2.48	4.15	8.59	35.65	175,000	229,900	6,450
52	2.36	4.08	8.50	34.68	142,000	207,000	5,970
54	2.60	4.05	8.49	34.38	144,000	175,700	5,110
57	2.50	4.16	9.04	37.61	282,000	653,000	17,360
60	2.45	4.25	8.92	37.91	138,000	686,000	18,100
63	2.49	4.07	8.70	35.41	185,000	216,100	6,100
69	2.57	4.10	8.50	34.85	153,000	180,600	5,180
75	2.58	4.14	8.58	35.52	190,000	259,900	7,320
81	2.37	4.20	8.54	35.87	148,000	219,800	6,130

* *Watertown Arsenal Report*, 1884.

The compressive strength here ranged from 5000 lb./in.^{2} to 18,000 lb./in.^{2} Average values for the strength of different kinds of brick in compression might be given as follows: soft brick, 900 lb./in.^{2}; hard brick, 3250 lb./in.^{2}; and vitrified brick, 17,500 lb./in.^{2} The latter includes paving brick.

Problem 310. The following bricks were tested in compression.

(*a*) Red face brick: sectional area, 28.45 in.^{2}; load at first crack, 379,000 lb.; load at maximum, 384,600 lb.

(*b*) Vitrified brick: sectional area, 27.46 in.^{2}; load at first crack, 72,000 lb.; load at maximum, 230,000 lb.

(*c*) Paving brick: sectional area, 26.72 in.^{2}; load at first crack, 51,000 lb.; load at maximum, 148,000 lb.

Find the intensity of stress at first crack and at maximum load in each case.

240. Modulus of elasticity of brick. As in the case of stone and concrete, the modulus of elasticity of brick in compression is not constant, but varies to some extent with the load. On account of this variation it is hard to give average values for the modulus of elasticity of brick, especially as the materials and methods of manufacture are so varied. Therefore in stating the modulus of elasticity it is also necessary to state the corresponding load. Strictly speaking, brick, stone, and concrete have no modulus of elasticity.

The table below is the result of a series of tests of dry-pressed and mud brick, tested edgewise in compression, and gives the modulus of elasticity for loads between 1000 lb./in.^{2} and 3000 lb./in.^{2}, and also at the highest stress observed.

MODULUS OF ELASTICITY FOR BRICK

Kind of Brick	Position in Kiln	Weight per Cubic Foot lb.	Modulus of Elasticity lb./in.^{2}		Compressive Strength lb./in.^{2}
			Between Loads of 1000 and 3000 lb./in.^{2}	At Highest Stress Observed	
Dry pressed	Top .	128.3	3,125,000	3,271,000	10,300
"	$\frac{1}{3}$ down . .	127.2	3,125,000	2,846,000	8,740
"	$\frac{2}{3}$ down . .	124.3	2,222,000	2,174,000	5,940
"	Bottom . .	119.8	1,205,000	. . . .	3,480
Mud . . .	Top . . .	144.3	10,000,000	8,654,000	19,170
"	$\frac{1}{4}$ down . .	136.4	7,692,000	7,576,000	15,670
"	$\frac{2}{3}$ down . .	130.6	5,263,000	4,545,000	10,420
"	Bottom . .	125.4	4,545,000	3,977,000	10,870

Problem 311. A dry-pressed brick of sectional area 9.72 sq. in. was tested in compression endwise. Measurements were taken on a gauged length of 5 in. and the following data obtained.

Applied Loads		In Gauged Length		Remarks
Total lb.	lb./in.^{2}	Compression in.	Set in.	
972	100	0.	0.	Initial load
1,944	200	.0003	0.	
3,888	400	.0007	0.	
5,832	600	.0012	0.	
7,776	800	.0015	0.	
9,720	1,000	.0017	0.	
11,664	1,200	.0020	. . . .	
13,608	1,400	.0024	. . . .	
15,552	1,600	.0028	. . . .	
17,496	1,800	.0030	. . .	
19,440	2,000	.0033	0.	
21,384	2,200	.0036	. . . . . .	
23,328	2,400	.0039	. . .	
25,272	2,600	.0042	. . .	
27,216	2,800	.0045	. . . .	
29,160	3,000	.0049	0.	E (1000 – 3000) = 3,125,000 lb./in.^{2}
31,104	3,200	.0051	. .	
33,048	3,400	.0054	. . . .	
34,992	3,600	.0057	. . . .	
36,936	3,800	.0060	. . . . . .	
38,880	4,000	.0063	0.	
40,824	4,200	.0066	. . .	
42,768	4,400	.0069	. . . .	
44,712	4,600	.0072	. . . .	
46,656	4,800	.0075	. . . .	
48,600	5,000	.0078	.0001	
50,544	5,200	.0081	. . .	
52,488	5,400	.0084	. . . .	
54,432	5,600	.0087	. .	
56,376	5,800	.0090	. . . .	
58,320	6,000	.0093	.0001	
60,264	6,200	.0097	. . . .	
62,208	6,400	.0100	. . . . .	
64,152	6,600	.0104	. . .	
66,096	6,800	.0107	. . .	
68,040	7,000	.0110	.0002	
69,984	7,200	.0113	. . . .	
71,928	7,400	.0117	. . . . . .	
73,872	7,600	.0120	. . . . . .	
75,816	7,800	.0124	. . .	
77,760	8,000	.0127	.0003	E (1000 – 8000) = 3,271,000 lb./in.^{2}
100,100	10,300	. .	. .	Ultimate strength

Draw the strain diagram. Compute the modulus of elasticity between loads of 1000 lb./in.^{2} and 3000 lb./in.^{2}, and also between 6000 lb./in.^{2} and 8000 lb./in.^{2}

241. Transverse tests of brick. Bricks are tested transversely by supporting them edgewise or flatwise upon two knife-edges and applying the load centrally by means of an ordinary testing machine. Care must be taken to provide suitable bearing surfaces for the knife-edges, in order to prevent local failure. In this test the upper fibers are in compression and the lower fibers in tension, and since brick is stronger in compression than in tension, failure is caused by rupture of the tension face. The fiber stress is computed from the formula

$$p = \frac{Ple}{4I},$$

where P is the breaking load in pounds, l is the length of span in inches, e is half the height, and I is the moment of inertia of a cross section. The fiber stress on the outer fiber at failure is usually called the **modulus of rupture.**

For paving brick the modulus of rupture varies from 1000 lb./in.^{2} to 3000 lb./in.^{2} For pressed brick, common brick, and medium brick the modulus of rupture varies from 300 lb./in.^{2} to 1200 lb./in.^{2}

The shearing strength of various grades of brick varies from 300 lb./in.^{2} to 2000 lb./in.^{2}

Problem **312.** A brick having a depth of 2.23 in. and a breadth of 3.95 in. was loaded centrally on a span of 6 in. The ultimate load was 1645 lb. Find the modulus of rupture.

242. Rattler test of brick. Paving bricks were formerly tested in abrasion in order to determine their ability to withstand wear. This test, however, does not approach the conditions of actual service, which consist of the impact of horses' feet as well as the abrasive action of traffic. To meet these conditions the rattler test was devised. The testing machine consists of a cast-iron barrel mounted horizontally, and the test is made by placing the brick, together with some harder material, such as cast iron, in the machine and revolving it at a certain speed for a certain length of time. The ratio of the amount of material broken or worn off in this way to the original weight of the brick put into the machine indicates the value of the brick in withstanding the conditions of service.

The **charge** usually consists of nine paving bricks or twelve other bricks, together with 300 lb. of cast-iron blocks, the volume of the

ricks being equal to about 8 per cent of the volume of the machine. 'he cast-iron blocks are of two sizes, the larger being about $2\frac{1}{2}$ in. quare and $4\frac{1}{2}$ in. long, with rounded edges and weighing at first $\frac{1}{2}$ lb. The smaller are about $1\frac{1}{2}$-in. cubes, with rounded edges. About 25 lb. of the smaller size and 75 lb. of the larger size are used; 800 revolutions are required, and must be made at the rate of about 0 per minute.*

During the first 600 revolutions the effect of the rattler action on he brick is to chip off the corners and edges. Thereafter the action s more nearly abrasive.

243. Absorption test of brick. A brick which absorbs a great mount of water is likely to be weakened and injured by frost. To neasure the amount of absorption, a dry brick is taken and a deter-nination of its absorbing capacity made, as in the case of stone Article 237).

Ordinary brick will absorb from 10 to 20 per cent of its own veight, and paving brick from 2 to 3 per cent.

This test is now little used, since a brick that fails in the absorp-ion test is of such poor quality that it will also fail when subjected o the crushing and cross-bending tests.

* See specifications of the National Brick Manufacturers' Association for rattler test.

CHAPTER XVI

TIMBER

244. Structure of timber. An examination of the cross section of a tree usually shows that it is made up of a rather dark interior core, or **heartwood**, and a lighter exterior portion, or **sapwood**, surrounded by the bark. In some species, such as the oaks, radial lines, called **medullary rays**, are seen running from the center toward the bark. If the cross section happens to be near a knot or other defect, this normal structure may be changed. If, however, no knots are present, a closer examination shows that both the sapwood and heartwood are made up of concentric rings, called **annual rings**, and that this appearance is due to a difference in structure. Part of the ring is seen to be denser than the rest, and, in fact, it is this difference in density which gives the section its characteristic appearance.

The annual rings in one stick of a certain species may be more widely separated than those in another stick of the same species, and the relative thickness of heartwood and sapwood may differ in different sticks. This indicates that the structure of timber varies considerably, and that therefore the physical properties also vary. This wide variation is seen in all substances found in nature, one instance of which has been shown in the case of natural stone. An investigation of the physical properties of such substances, therefore, is more difficult than that of a more homogeneous substance. However, the extensive use of timber as a structural material makes a knowledge of its structure and properties of the utmost importance.

245. Annual rings. Each of the concentric rings in timber represents the growth of one year. The inner or less dense portion represents the more rapid spring growth, while the outer dense portion represents the slower summer and fall growth. The number of rings per inch indicates the rate of growth for that number of years. If the number of rings per inch be few, the growth has been rapid and

ıe spring growth predominates, making the wood somewhat weak. ſ, on the contrary, the number of rings per inch be many, a slow rowth is indicated and there is a greater amount of the dense, strong ımmer and fall wood. The number and character of the annual ngs may thus give some idea of the strength of a piece of timber.

246. Heartwood and sapwood. The heartwood of a tree may be onsidered a lifeless conical core, which is increased each year by the ddition of a portion of the outer sapwood. Both the sapwood and eartwood contain small tubes that extend from the roots of the tree ɔ the branches. These tubes in the sapwood carry water charged rith nourishment to the branches and growing parts of the tree. In he heartwood the tubes no longer act as conveyors, although they till contain moisture. The heartwood is the mature wood and is ıore valuable for structural purposes.

247. Effect of moisture. It is well known that green wood is not s strong as the same wood when seasoned, which indicates that the ffect of moisture in timber is to lessen its strength. A live tree as ; stands in the forest contains a great deal of moisture. When it as been cut, sawed, and dried, most of this moisture has evaporated, ut considerable still remains, and however well seasoned timber may e, it will still contain some moisture.

In making tests of timber, therefore, it is necessary to determine he percentage of moisture in order that the results may be compared rith the results of other tests. This is determined by cutting a mall piece from the uninjured portion of the test piece and weighing efore and after thorough drying. The difference in weight divided y the dry weight and multiplied by 100 gives the percentage of ıoisture.

248. Strength of timber. The strength of timber depends upon the mount of heartwood or sapwood, knots (sound or loose), wind shakes nd checks, cracks, or any defect that breaks the continuity of the ber. In general, the strength of timber is indicated by its weight, he heaviest timbers being the strongest. Timber is strongest along the rain both in tension and compression, as will be seen in what follows.

It has been found that values obtained for the strength of timber y testing small, carefully selected test pieces are much higher than hose obtained by testing large commercial timbers. This is what

might be expected, since the larger commercial pieces contain knots and other defects not found in the selected test pieces. It has been found also that the place and conditions of growth, time of felling, method and time of seasoning, and many other factors have each some effect upon the strength of timber. Since the weight of timber is an indication of its strength, some idea of the relative strength of the more common species may be obtained by referring to the table given below.*

WEIGHT OF KILN-DRIED WOOD OF DIFFERENT SPECIES

	Approximate		
	Specific Weight	Weight of 1 cu. ft. lb.	Weight of 1000 ft. of Lumber lb.
(*a*) Very heavy woods:			
Hickory, oak, persimmon, osage orange, black locust, hackberry, blue beech, best of elm, ash	0.70–0.80	42–48	3700
(*b*) Heavy woods:			
Ash, elm, cherry, birch, maple, beech, walnut, sour gum, coffee tree, honey locust, best of Southern pine, tamarack	0.60–0.70	36–42	3200
(*c*) Woods of medium weight:			
Southern pine, pitch pine, tamarack, Douglas spruce, western hemlock, sweet gum, soft maple, sycamore, sassafras, mulberry, light grades of birch and cherry	0.50–0.60	30–36	2700
(*d*) Light woods:			
Norway and bull pine, red cedar, cypress, hemlock, the heavier spruce and fir, redwood, basswood, chestnut, butternut, tulip, catalpa, buckeye, heavier grades of poplar . .	0.40–0.50	24–30	2200
(*e*) Very light woods:			
White pine, spruce, fir, white cedar, poplar .	0.30–0.40	18–24	1800

249. **Compression tests.** Compression tests are made on short blocks and long columns. For the short-block test the piece is placed in an ordinary testing machine between the moving head and the platform;

*Bureau of Forestry, *Bulletin No. 1 D*, "Timber."

with its ends as nearly parallel as possible, and the compression is measured by an ordinary compressometer, or similar instrument for measuring the lowering of the moving head. To provide for the non-parallelism of the ends, it is well to use a spherical bearing for one of the bearing ends. This will insure the proper "lining up" of the specimen so that the compression will be along the grain.

A strain diagram may be drawn by plotting loads in lb./in.^{2} as ordinates and the corresponding relative compressions as abscissas. The elastic limit, modulus of elasticity, modulus of resilience, and maximum strength may then be obtained from the diagram in the usual manner. Failure is either due to a splitting of the specimen or to a shearing off at an angle of about 30° to the horizontal (Fig. 190). The latter is the characteristic failure for green timber.

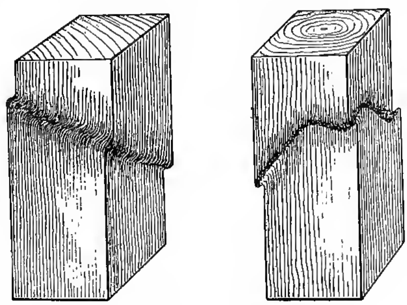


FIG. 190

The tests on long columus are made in much the same way as the tests on short blocks. Provision is made for fixing the ends of the columns so as to give the standard end conditions, namely, square ends, round ends, pin and square ends, etc. In either case sufficient data is taken to get a load-deflection curve by measuring the deflections at the center corresponding to selected load increments. These deflections are usually measured in two directions at right angles to each other.*

Problem 313. Fig. 191 represents the results of compression tests of pine, poplar, and oak, plotted with loads in pounds as ordinates and compression in inches as abscissas. The blocks were all 7 in. high, with an area of cross section as follows: pine, 2 in. × 1.48 in.; poplar, 2 in. × 1.48 in.; oak, 2 in. × 1.47 in. Redraw the curves, plotting the loads in lb./in.^{2} as ordinates and the corresponding unit compressions in inches as abscissas. Determine for each material the elastic limit, the modulus of elasticity, and the modulus of elastic resilience. Also compare the results obtained with the results reported for these materials in compression in the following tables.

* For a report of the tests that have been made on full-sized timber columns the student is referred to Lanza's *Applied Mechanics*.

250. Flexure tests. Flexure tests are usually made by supporting a rectangular piece at both ends and loading it in the middle, care being taken to guard against local failure at the supports and at the point of application of the load. This local failure may be prevented by inserting some kind of metal plate between the beam and the knife-edge. The deflections of the beam for specified loads are measured by means of a deflectometer, usually measuring to .01 in. or .001 in. From the data obtained from a test, a strain diagram may be drawn by plotting loads in pounds as ordinates and deflections in inches as abscissas. The fiber stress for any load within the elastic limit is determined, for central loading, from the formula

$$p = \frac{Ple}{4I};$$

and the modulus of elasticity from the formula (Article 67)

$$E = \frac{Pl^3}{48DI}.$$

The formulas used to determine the fiber stress in the case of the flexure of beams $\left(\frac{pI}{e} = M_{\max}\right)$ are true only within the elastic limit of the material. They are used, however, to determine the fiber stress beyond the elastic limit, although they are only approximately true beyond this limit. The value of the fiber stress at rupture as determined by the formula is usually designated as the **modulus of rupture** (Article 65); it is expressed in lb./in.^{2}

On account of the peculiar structure of timber the character of the fracture due to a failure in flexure is rather difficult to predict. In case the specimen is free from knots and the grain is parallel to the length of the piece, failure from concentrated central loading is likely to take place either on the tension or the compression side, or both. It may happen, however, even in the case of such a perfect specimen as indicated, that failure will be due to horizontal shear. In such cases shearing takes place along the spring growth of one of the annual rings. This may have been weakened previously by wind shakes.

If part of the beam is sapwood and part heartwood, the fracture will be influenced thereby, due to the difference in the strength of the two portions. A cross grain may cause a failure due to splitting.

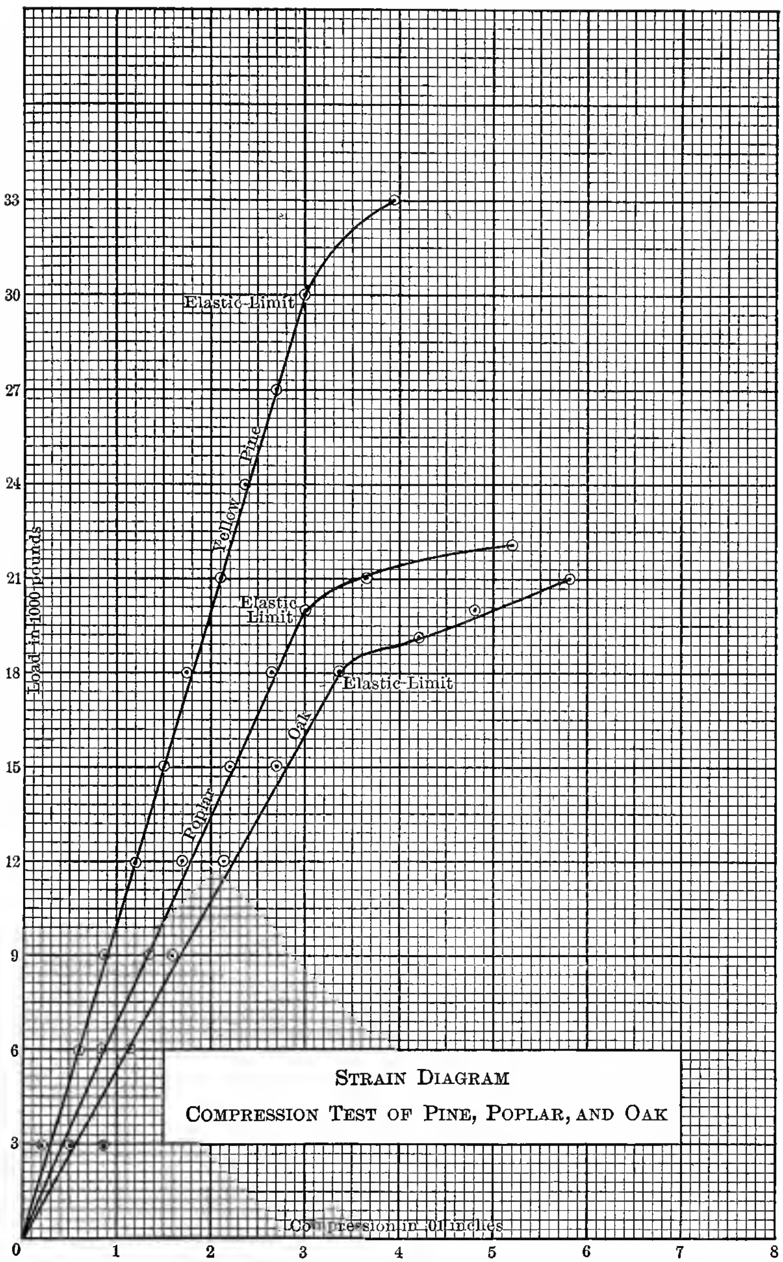

Strain Diagram
Compression Test of Pine, Poplar, and Oak
Load in 1000 pounds
Compression in .01 inches
Elastic Limit
Elastic Limit
Elastic Limit
Yellow Pine
Poplar
Oak
33
30
27
24
21
18
15
12
9
6
3
0
1
2
3
4
5
6
7
8

Fig. 191

Knots of any kind near the central portion of the beam may determine the fracture and cause the beam to break off almost squarely. No law has yet been determined which will give the effect of knots of various sizes on the strength of timber.

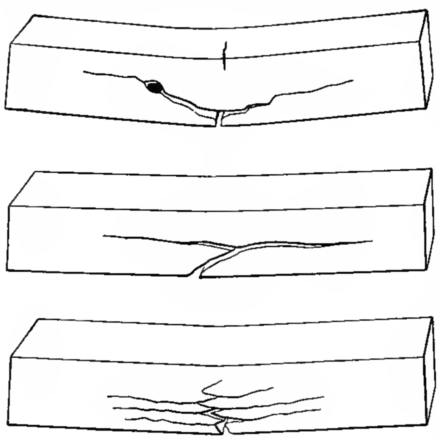


Fig. 192

Some characteristic failures in flexure are shown in Fig. 192. The lower beam shows a normal failure on the tension side. The two upper beams show the fracture of a somewhat more brittle material, the fracture being influenced by the presence of knots. The upper beam also shows a compression failure.

Problem 314. A rectangular pine beam, width 1.48 in., height 1.99 in., and span 30 in., was tested in flexure by being supported at both ends and loaded in the middle, and the following data obtained. Draw the strain diagram, plotting loads in pounds as ordinates and deflection in inches as abscissas. Locate the elastic limit and compute the fiber stress on the outer fiber at the elastic limit. Also compute the modulus of rupture, the modulus of elasticity, and the modulus of elastic resilience.

Central Load lb.	Deflection at Center in.	Central Load lb.	Deflection at Center in.
100	.034	900	.305
200	.061	1000	.341
300	.097	1100	.393
400	.132	1200	.451
500	.166	1300	.583
600	.201	1400	.670
700	.234	1500	.810
800	.270	1585 (maximum)	.952

251. Shearing tests. The shearing strength of timber parallel to the grain is usually measured by finding the force necessary to cause a small projecting block of the material to shear off along the grain.

In this case the line of action of the force is parallel to the grain. The intensity of stress is obtained by dividing the force by the area of the sheared surface.

252. Indentation tests. Indentation tests are intended to show the crushing strength of timber perpendicular to the grain. A rectangular piece of the timber is usually chosen, and a metal block whose width equals the width of the specimen is pressed into it by an ordinary testing machine. Convenient load increments are taken, and these, together with the corresponding compressions, give sufficient data for a load compression curve from which the elastic properties may be determined. Fig. 193 illustrates a specimen that has been tested in compression perpendicular to the grain.

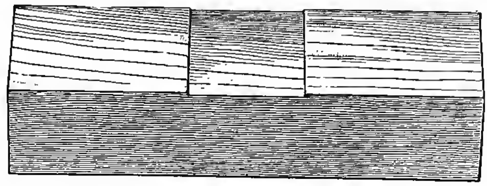


Fig. 193

253. Tension tests. Tension tests of timber are seldom made on account of the difficulty of obtaining satisfactory test pieces. The specimens to be tested must be much larger at the ends than in the middle in order to provide for attachment in the heads of the testing machine, and for this reason the piece is likely to fail by the shearing off of the enlarged ends, or by the pulling out of the fastenings. This test, therefore, is little used, the flexure test being relied upon to furnish information regarding the tensile strength of timber.

254. European tests of timber. As early as the middle of the eighteenth century tests to determine the strength of timber were made in France. This work was done for the most part from a scientific standpoint. The most important European tests were carried out by Bauschinger in his laboratory at Munich, from 1883 to 1887. The object of these tests was to determine the effect of the *time of felling* and *conditions of growth* upon the strength of Scotch pine and spruce. From these tests Bauschinger drew the following conclusions.

1. Stems of spruce or pine which are of the same age at equal diameters, and in which the rate of growth is about equal, have the same mechanical properties (when reduced to the same moisture contents), irrespective of local conditions of growth.

2. Stems of spruce or pine which are felled in winter have, when tested two or three months after the felling, about 25 per cent greater strength than those felled in summer, other conditions being the same.

He notes, however, that later tests may change these conclusions somewhat.

AVERAGE RESULTS OF TIMBER TESTS MADE FOR THE TENTH CENSUS

Name of Species	Transverse Tests		Compression Tests	
	Modulus of Rupture lb./in.^{2}	Modulus of Elasticity lb./in.^{2}	Compression Parallel to Grain lb./in.^{2}	Compression Perpendicular to Grain lb./in.^{2}
Poplar	9,400	1,330,000	5000	1120
Basswood	8,340	1,172,000	5190	880
Ironwood	7,540	1,158,000	5275	2000
Sugar maple	16,500	2,250,000	8800	3600
White maple	14,640	1,800,000	6850	2580
Box elder	7,580	873,000	4580	1580
Sweet gum	9,330	1,300,000	6630	1880
Sour gum	12,200	1,262,000	6780	2780
White ash	10,000	1,200,000	7540	2250
Black walnut	11,900	1,560,000	8000	2680
Slippery elm	12,450	1,445,000	7670	2330
Sycamore	7,000	790,000	6400	2700
Hickory (shellbark)	16,600	1,912,000	8360	3500
White oak	11,770	1,300,000	7100	3100
Red oak	15,100	1,640,000	6800	2700
Black oak	14,900	1,450,000	6960	2980
White pine	8,100	1,225,000	4800	1050
Yellow pine	11,100	1,400,000	5580	1450
Loblolly pine	12,250	1,567,000	6100	1520
Long-leaved pine	15,450	1,901,000	8300	1920
Hemlock	9,480	1,138,000	5400	1100
Red fir	13,270	1,870,000	7780	1750
Tamarack	13,150	1,917,000	7400	1480
Red cedar	11,800	938,000	6300	2000
Cottonwood	10,440	1,450,000	5000	1100
Beech	16,200	1,730,000	6770	2840
Averages of all species given above	11,800	1,445,000	6600	2110

255. Tests made for the tenth census. In the United States, tests were made for the tenth census on four hundred and twelve species of timber. The test specimens were all small, selected pieces, 1.57 in. × 1.57 in. in cross section, and 43 in. long, and were seasoned in a dry, cool building for two years. On account of the number of species tested the results obtained are not conclusive, but should be taken as indicating the probable values for the strength of the timbers tested. On page 344 is given a table of the averages for some of the species tested. Since the test pieces were all small, selected specimens, the results are probably higher than would have been obtained from larger commercial specimens.

In the transverse tests the specimens were supported at both ends and loaded in the middle, the span being 39.37 in. The compression tests parallel to the grain were made on pieces 1.57 in. × 1.57 in. in cross section, and 12.6 in. long. Indentation tests were made on pieces 1.57 in. × 1.57 in. in cross section and 6.3 in. long. The test pieces in the latter case rested upon the platform of the testing machine, and the tests were made by crushing perpendicular to the grain with a plate 1.57 in. × 1.57 in. in size, by lowering the moving head of the machine.

256. Tests made by the Bureau of Forestry. The most extensive series of timber tests that has ever been undertaken has been begun by the United States Department of Agriculture under the direction of the Bureau of Forestry. These tests were begun in 1891, under the direction of Professor J. B. Johnson, at St. Louis. Thirty-two species were tested and 45,000 tests were made. The material was selected with special reference to the conditions under which the trees were grown, and the test pieces were small, selected specimens. The table on page 346 gives the average results of some of the tests.*

In the table the results have been reduced to an amount of moisture equivalent to 12 per cent of the dry weight.

A comparison of this table with that of the tenth census shows as close an agreement in most cases as might be reasonably expected when the variability of timber is considered, and serves to extend and verify the results of the previous work.

* U. S. Forestry Circular, No. 15.

RESULTS OF TIMBER TESTS MADE BY THE UNITED STATES BUREAU OF FORESTRY

Species	Transverse Tests		Compression Tests		Shearing Tests
	Modulus of Rupture lb./in.^{2}	Modulus of Elasticity lb./in.^{2}	Compression Parallel to Grain lb./in.^{2}	Compression Perpendicular to Grain lb./in.^{2}	Shearing along the Grain lb./in.^{2}
Long-leaf pine	12,600	2,070,000	8,000	1260	835
Cuban pine	13,600	2,370,000	8,700	1200	770
Short-leaf pine	10,100	1,680,000	6,500	1050	770
Loblolly pine	11,300	2,050,000	7,400	1150	800
White pine	7,900	1,390,000	5,400	700	400
Red pine	9,100	1,620,000	6,700	1000	500
Spruce pine	10,000	1,640,000	7,300	1200	800
Bald cypress	7,900	1,290,000	6,000	800	500
White cedar	6,300	910,000	5,200	700	400
Douglas spruce	7,900	1,680,000	5,700	800	500
White oak	13,100	2,090,000	8,500	2200	1000
Overcup oak	11,300	1,620,000	7,300	1900	1000
Post oak	12,300	2,030,000	7,100	3000	1100
Cow oak	11,500	1,610,000	7,400	1900	900
Red oak	11,400	1,970,000	7,200	2300	1100
Texan oak	13,100	1,860,000	8,100	2000	900
Yellow oak	10,800	1,740,000	7,300	1800	1100
Water oak	12,400	2,000,000	7,800	2000	1100
Willow oak	10,400	1,750,000	7,200	1600	900
Spanish oak	12,000	1,930,000	7,700	1800	900
Shagbark hickory	16,000	2,390,000	9,500	2700	1100
Mockernut hickory	15,200	2,320,000	10,100	3100	1100
Water hickory	12,500	2,080,000	8,400	2400	1000
Bitternut hickory	15,000	2,280,000	9,600	2200	1000
Nutmeg hickory	12,500	1,940,000	8,800	2700	1100
Pecan hickory	15,300	2,530,000	9,100	2800	1200
Pignut hickory	18,700	2,730,000	10,900	3200	1200
White elm	10,300	1,540,000	6,500	1200	800
Cedar elm	13,500	1,700,000	8,000	2100	1300
White ash	10,800	1,640,000	7,200	1900	1100
Green ash	11,600	2,050,000	8,000	1700	1000
Sweet gum	9,500	1,700,000	7,100	1400	800

The effect of the presence of moisture on the strength of timber was also investigated when these tests were made by testing some of

the foregoing species endwise in compression while green. The following table gives the results of these tests in lb./in.^{2} The pieces contained over 40 per cent of moisture. A comparison of the results obtained from these tests with those reported in the preceding table shows that the compressive strength has been diminished from 50 to 75 per cent by the presence of the given percentage of moisture.

COMPRESSIVE TESTS OF GREEN TIMBER

SPECIES	NUMBER OF TESTS	HIGHEST SINGLE TEST lb./in.^{2}	LOWEST SINGLE TEST lb./in.^{2}	AVERAGE OF ALL TESTS lb./in.^{2}
Long-leaf pine	86	7300	2800	4300
Cuban pine	38	6100	3500	4800
Short-leaf pine	8	4000	3000	3300
Loblolly pine	69	5500	2600	4100
Spruce pine	71	4700	2800	3900
Bald cypress	280	8200	1800	4200
White cedar	34	3400	2300	2900
White oak	25	7000	3200	5300
Overcup oak	45	4900	2800	3800
Cow oak	58	4900	2300	3800
Texan oak	39	6000	3100	5200
Willow oak	49	5500	2300	3800
Spanish oak	52	5100	2500	3900
Shagbark hickory	22	6900	3500	5700
Mockernut hickory	18	7200	4500	6100
Water hickory	4	5600	4700	5200
Nutmeg hickory	26	5500	3700	4500
Pecan hickory	4	3800	3300	3600
Pignut hickory	5	6200	4700	5400
Sweet gum	6	3600	3000	3300

Certain special tests were also made to determine:

(*a*) The effect of *bleeding* (tapping for turpentine) on long-leaf pine.

(*b*) Influence of size on transverse strength of beams.

(*c*) Influence of size on compressive strength.

(*d*) The effect of hot-air treatment in dry kilns on strength.

The results obtained from these tests indicated:

(*a*) That bleeding does not affect the strength.

(*b*) That large, sound beams *may* be as strong as small ones cut from the same piece; that is, large beams may show the same fiber stress as small ones.

(*c*) That large, sound pieces in compression *may* be as strong as small ones cut from the same piece; that is, the intensity of compressive stress may be the same.

(*d*) That there were no detrimental effects.

The results of the tests made by the Bureau of Forestry, as outlined in this article, should not be taken as conclusive, since not a sufficient number of tests were made to establish values. The pieces were in most cases small, and specially selected, and the results are of more value from a scientific than from a commercial standpoint, since the lumber of commerce contains knots, wind shakes, and other defects that lessen its strength.

257. Recent work of the United States Forest Service. The United States Forest Service (formerly known as the Bureau of Forestry) has recently made extensive studies of the uses and durability of the various commercial woods of the United States, and has also conducted a series of tests to determine their strength, the most important of which are as follows:

(*a*) Tests of commercial-size beams of various timbers found on the market to determine

1. The effect of knots and other defects on the strength.
2. The effect of moisture on the strength.
3. The effect of preservatives on the strength.
4. The effect of methods of seasoning on the strength.

(*b*) Tests of materials used in the construction for vehicles for such purposes as spokes, axles, and poles.

(*c*) Tests of the strength of packing boxes.

(*d*) Tests of the strength of railroad ties.

In each of these investigations one of the objects has been to determine, if possible, some so-called inferior woods that might be used in place of varieties that are superior but are becoming scarce. The test pieces for (*a*) were large commercial pieces in which knots and other defects occur, as they do in the structural timbers used by engineers.

A summary of some of the cross-bending tests is given in the following table.

FLEXURE TESTS OF COMMERCIAL TIMBER

Species	Grade	Average Number of Sticks	Time Seasoned months	Moisture per cent	Weight per Cubic Foot: As Tested lb.	Weight per Cubic Foot: Dry lb.	Modulus of Rupture lb./in.^{2}	Modulus of Elasticity lb./in.^{2}
Red fir:								
Shipments A and C	Selects	22	6 to 12	22.6	37.1	30.2	8810	1,925,000
	Merchantable	29		20.8	34.5	28.4	7730	1,825,000
	Seconds	16		19.5	31.9	26.7	6290	1,630,000
Shipment B	Selects	14	3	27.6	30.9	24.2	6250	1,280,000
	Merchantable	15		26.5	33.7	26.6	5340	1,320,000
	Seconds	25		26.2	35.1	27.8	4280	1,400,000
Shipments A, B, and C	Selects	36	. .	24.5	34.7	27.9	7780	1,675,000
	Merchantable	44		22.7	34.8	28.4	6920	1,650,000
	Seconds	41		23.6	33.8	27.4	5070	1,490,000
Average of shipments A, B, and C	All grades	121	. . .	23.6	33.4	27.8	6580	1,570,000
Western hemlock	All grades	30	3 to 6	32.2	35.4	26.8	5565	1,260,000
North Carolina loblolly pine	Square edge	20	3	37.2	42.8	31.2	6187	1,479,000
Long-leaf pine	Merchantable	26	6 to 12	26.7	53.3	42.1	8210	1,790,000

The grades **selects, merchantable,** and **seconds,** referred to in the table, are those from the Pacific Coast standard grading rules for Douglas Fir for 1900. A copy of these rules is here given.

Merchantable. This grade shall consist of sound, strong lumber, free from shakes, large, loose, or rotten knots, and defects that materially impair its strength; it shall be well manufactured and suitable for good substantial constructional purposes.

Will allow occasional variations in sawing or occasional scant thicknesses, sound knots, pitch seams, and sap on corners, one third the width and one half the thickness. Defects in all cases to be considered in connection with the size of the piece and its general quality.

Seconds. This grade shall consist of lumber having defects which exclude it from grading as merchantable.

Will allow knots and defects which render it unfit for good substantial constructional purposes, but suitable for an inferior class of work.

Selects. Shall be sound, strong lumber, good grain, well sawn.

Will allow, in sizes 6 by 6 and less, knots not to exceed 1 in. in diameter; sap on corners one fourth the width and one half the thickness; small pitch seams when not exceeding 6 in. in length.

COMPARATIVE STRENGTH OF LARGE AND SMALL SPECIMENS

(For other work done by the Bureau of Forestry the student is referred to the bulletins giving the results of timber tests)

Species	Compression Parallel to Grain			Cross Bending				
	Dimensions	Average		Section	Span	Average		
		At Elastic Limit lb./in.²	At Rupture lb./in.²			Fiber Stress at Elastic Limit lb./in.²	Modulus of Rupture lb./in.²	Modulus of Elasticity lb./in.²
Red fir, A	5″ × 5″ × 30″ 2″ × 2″ × 10″	3550 3300	4815 4906	6″ × 8″ 2″ × 2″	16′ 24″	5740 6840	7,775 10,220	1,835,000 1,644,000
Ratio of small sticks to large		.935	1.02			1.17	1.315	.898
Red fir, A	5″ × 5″ × 30″ 2″ × 2″ × 10″	3650 3720	4740 5140	8″ × 16″ 6″ × 8″ 2″ × 2″	16′ 7′ 24″	5450 5840 6810	7,830 9,060 10,320	1,824,000 1,572,000 1,658,000
Ratio of small sticks to large		1.02	1.09			1.25	1.315	.91
Red fir, C	5″ × 8″ × 30″ 2″ × 2″ × 10″	3120 2360	4235 4540	8″ × 16″ 5″ × 8″ 2″ × 2″	16′ 7′ 24″	4720 4930 5724	6,850 7,899 9,110	1,517,000 1,331,000 1,432,000
Ratio of small sticks to large		.95	1.07			1.21	1.33	.95
Red fir, C	5″ × 8″ × 30″ 2″ × 2″ × 10″	4170 4050	5090 5293	5″ × 8″ 2″ × 2″	16′ 24″	5360 6320	8,030 9,780	1,850,000 1,600,000
Ratio of small sticks to large		.985	1.04			1.14	1.22	.865
Western hemlock, Oregon	5″ × 5″ × 30″ 2″ × 2″ × 10″	2556 2860	3353 3841	8″ × 16″ 6″ × 8″ 2″ × 2″	16′ 7′ 24″	3870 4030 4950	5,115 5,742 7,871	1,308,000 1,238,000 1,314,000
Ratio of small sticks to large		1.15	1.14			1.28	1.54	1.005
Loblolly pine, Virginia	4″ × 8″ × 16″ 2″ × 2″ × 6″	1766	2785 3216	4″ × 8″ 2″ × 2″	10′ 30″	2078 3038	5,101 6,719	1,192,000 1,054,000
Ratio of small sticks to large			1.15			1.46	1.34	.885
Loblolly pine, Virginia	8″ × 8″ × 20″	1346	1988	8″ × 8″ 2″ × 2″	6′–15′ 30″	1847 1868	3,281 3,916	729,000 550,000
Ratio of small sticks to large						1.01	1.19	.755
Loblolly pine, South Carolina	4″ × 8″ × 16″ 4″ × 8″ × 16″ 2″ × 2″ × 6″	3123 2373	4396 3528 3635	8″ × 14″ 8″ × 14″ 4″ × 8″ 4″ × 8″ 2″ × 2″	15′–6″ 15′–6″ 6′–6″ 6′–6″ 30″	3953 3268 3869 3443 4050	6,336 6,039 8,099 7,090 7,890	1,534,000 1,420,000 1,542,000 1,329,000 1,456,000
Ratio of small sticks to large			.917			1.12	1.275	.985
Long-leaf pine, Georgia				10″ × 12″ 4″ × 5″	16′ 7′	5530 6200	8,280 9,290	1,807,000 1,801,000
Ratio of small sticks to large						1.12	1.12	.99
Ratio of small sticks to large based on total number of tests		1 1.01	1 1.06			1 1.19	1 1.29	1 .918

In sizes over 6 by 6, knots not to exceed 2 in. in diameter, varying according to the size of the piece; sap on corners not to exceed 3 in. on both face and edge; pitch seams not to exceed 8 in. in length.

Defects in all cases to be considered in connection with the size of the piece and its general quality.

The cross-bending tests of 1 were made upon large specimens ranging in size from 6 in. × 8 in. × 7 ft., to 8 in. × 16 in. × 16 ft. The table shows that the modulus of rupture is less for the poorer grades of timber than for the selects, showing the effect of knots and other imperfections. The modulus of elasticity, indicating the stiffness, is less for the poorer grades, except in the case of shipment B of red fir.

The same report also makes a comparison of the strength of large sticks and small sticks, both in cross bending and in compression parallel to the fiber.

The table on page 350 gives average values obtained from this report, and indicates that the strength of the small sticks is, in nearly every case, greater than the strength of the large sticks. The modulus of elasticity is less for the small sticks than for the large ones, indicating a greater stiffness for the latter.

258. Treated timber. The increasing scarcity of good timber and the consequent rise in price has called the attention of American engineers to the necessity for the use of preservatives in order to lengthen the life of the timber for commercial purposes. This has developed a new branch of engineering in this country, based on the use of many things learned by the Europeans, who were the originators of some of the best methods of treatment.

When the tree is cut down and the timber seasoned (dried), a portion of the water evaporates from the sap, leaving the food materials deposited upon the cell walls. These materials are excellent food for bacteria and various forms of fungi that cause early decay of the timber if allowed to carry on their destructive work. In the early days, when timber was plentiful, no attempt was made to preserve wood from the destructive action of bacteria, but with increasing scarcity of good timber various methods of treatment have been devised. The simplest method, of course, is the application of common paint. This closes all the pores and protects the wood from the action of bacteria, but this method cannot be made use of where the

timber is in or near the ground or water, since the continued moisture causes the paint to peel off. The methods most generally used for treating timber for commercial purposes are given in the following paragraphs.

Zinc chloride process. The zinc chloride process is the cheapest, and until the last few years the one most widely used in this country. It consists of impregnating the wood fibers with a solution containing about one half a pound of dried zinc chloride per cubic foot of timber. The treatment is carried out as follows: Air-seasoned timber or timber that has been steamed to drive off the moisture is placed in a cylinder; a vacuum is then maintained, while the solution is being introduced, until the timber is covered. Pressure is then applied up to 100 to 125 lb./in.^{2} by pumping in additional solution. When the penetration has been sufficient, the solution is drained off. The principal difficulty with the timber treated by this process comes from the injury caused by steaming and the subsequent rapid leaching out of the zinc chloride. This treatment requires about seven hours.

Absorption process. In this process of treatment and those that follow the preservative used is creosote oil. This oil is obtained from coal tar, a by-product of artificial gas manufacture and the coke ovens. The creosote oil is distilled from coal tar at temperatures between 240° and 270° C. This absorption process is also known as a non-pressure process. Air-dried timber is placed in a receptacle and covered with the boiling preservative. This boiling tends to expel some moisture from the wood. After boiling, the excess creosote is drained off and the timber is immersed in cold preservative. In this way greater absorption is obtained on account of differences in temperature and pressure. This process is used principally for butts of telegraph poles, fence posts, and ties, in limited numbers. About 6 to 12 lb. of creosote oil per cubic foot may be absorbed by this process. The time required for treatment varies from seven to fourteen hours.

Full-cell creosoting process. The seasoned timber, which may be steamed to reduce moisture and expel sap, is placed in a vacuum and creosote introduced until the timber is submerged. A pressure of 100 to 125 lb./in.^{2} is then maintained, forcing the creosote into the wood. The creosote is then drained from the tank and, generally, a low vacuum is maintained to draw out the excess preservative.

An absorption of as much as 20 lb. of creosote oil per cubic foot of timber is possible by this process. The time required, including steaming, is about seven hours.

Rüping process. When this treatment is used, compressed air is forced into the pores of the wood, and while under this compression, creosote oil is introduced under a higher pressure (150 lb./in.^{2}). When the pressure is relieved and the creosote drained off, a vacuum is produced, allowing the compressed air in the pores of the wood to expand and force out the excess creosote. This leaves about 4 to 6 lb. of creosote per cu. ft. of timber. The cell walls are left lined with the preservative, whereas in the full-cell process the cells themselves are left nearly full. The Rüping process is accordingly much more economical in the use of cresote. The time required for this treatment is about four hours.

259. Strength of treated timber. The question naturally arises as to whether or not the treatment to which timber is subjected in introducing the preservative has any effect upon its strength in tension, bending, compression, and shear. The question as to whether or not the preservative itself weakens the timber must also be considered. To answer these questions the United States Forest Service has made an extended study of the strength of treated timber. The results of some of these tests are shown in the following tables.

Southern-Pine Bridge Stringers, Treated and Untreated Static Bending, ⅓ Point Loading Nominal Size, 8 in. × 16 in. × 14 in.						Compression Perpendicular to Grain Nominal Size 8 in. × 16 in. × 30 in.	
		Natural		Treated		Natural	Treated
Species	Moisture Condition	Deflection at Maximum Load in.	Modulus of Rupture lb./in.^{2}	Deflection at Maximum Load in.	Modulus of Rupture lb./in.^{2}	Fiber Stress at Elastic Limit lb./in.^{2}	Fiber Stress at Elastic Limit lb./in.^{2}
Pines: Long-leaf	air dry	2.14	6466	1.64	6376	741	793
Loblolly	air dry	1.76	6392	1.47	5380	653	461
Long-leaf	partially air dry	1.80	5151	1.90	5132	719	586
Loblolly	partially air dry	1.46	4858	1.60	4150	574	352

BRIEF SUMMARY OF RESULTS OF TESTS ON TREATED TIES

Strength of Full-sized Ties in Rail Bearing (Compression Perpendicular to Grain)	Compressive Strength at Elastic Limit lb./in.²	Pulling Resistance of Spikes		Lateral Resistance of Spikes Load at ¼-in. Displacement lb.
		Common lb.	Screw lb.	
Natural red oak ties	1093	8026	13,855	
Burnettized red oak ties	1065	7826	13,781	
The above ties were from Carbondale, Ill.				
Natural red oak ties	1239	8935	14,686	
Creosoted red oak ties (Lowry)	1285	8303	14,522	
The above red oak ties were from Shirley, Ind., and were drier than those from Carbondale, Ill.				
Natural red oak ties	1060	6792	11,418	4028
Treated red oak ties (Rüping)	978	6299	10,962	4211
Natural red oak ties	962	6805	12,521	4610
Treated red oak ties (Full Cell)	999	6853	11,671	4458
The above ties were from Somerville, Tex.				
Natural loblolly pine ties	510	3720	8,003	
Treated loblolly pine ties (Rüping)	503	4583	7,787	
Natural loblolly pine ties	479	3621	8,200	
Treated loblolly pine ties (Lowry)	499	3407	7,936	
The above loblolly pine ties were from Grenada, Miss.				
Natural loblolly pine ties	619	3660	8,040	2831
Treated loblolly pine ties (Rüping)	696	3980	9,020	3486
Natural loblolly pine ties	730	4755	9,012	2979
Treated loblolly pine ties (Full Cell)	729	3986	8,747	3830
Natural loblolly pine ties	721	3894	8,911	3200
Treated loblolly pine ties (Crude Oil)	529	2069	7,495	2875
The above loblolly pine ties were from Somerville, Tex.				
Natural short-leaf pine ties	799	4624	11,136	3525
Treated short-leaf pine ties (Rüping)	823	4626	9,684	3645
Natural short-leaf pine ties	609	4387	9,714	3254
Treated short-leaf pine ties (Full Cell)	659	4532	9,805	3417
Natural short-leaf pine ties	517	4068	9,182	3372
Treated short-leaf pine ties (Crude Oil)	373	1816	7,182	3439
Natural long-leaf pine ties	677	4465	9,001	3255
Treated long-leaf pine ties (Rüping)	737	4458	9,170	3276
Natural long-leaf pine ties	703	3445	8,474	3258
Treated long-leaf pine ties (Full Cell)	712	3634	9,290	3542
Natural red gum ties	916	4383	10,010	3789
Treated red gum ties (Rüping)	884	4490	9,720	3861
Natural red gum ties	843	3650	9,565	3765
Treated red gum ties (Full Cell)	833	3815	9,205	3679
Natural red gum ties	731	3615	9,885	3450
Treated red gum ties (Crude Oil)	655	2540	9,750	3500
The above short-leaf and long-leaf pines and red gum ties were from Somerville, Tex.				

An examination of the results of tests of the bridge stringers shows that there is little decrease in strength due to the action of the creosote in the case of the air-dry, long-leaf pine. The loblolly pine, air dried, shows a decrease in strength of 16 per cent in bending and 29 per cent iu compression. The long-leaf pine, partially air dried, shows no appreciable decrease in strength in bending, but about 18 per cent decrease in compression. Loblolly pine, partially air dried, shows 14 per cent decrease in bending strength and 38 per cent in compressive strength. These tests seem to show that long-leaf pine is injured very little, if any, by the creosote, while loblolly pine is injured appreciably. Treated oak ties (results not given here) show a decrease iu strength of from 5 to 10 per cent. Douglas fir and Wisconsin white pine show little or no effect due to treatment so far as bending and compression are concerned, but show a decrease in strength of from 20 to 25 per cent in shear.

CHAPTER XVII

ROPE, WIRE, AND BELTING

260. Wire. Wire is made from a steel or iron rod by pulling it through a hole, or **die**, of smaller diameter than the rod. This is called **drawing**, and is done while the metal is cold. It is known as **wet drawing** when the metal is lubricated, and as **dry drawing** when no lubricant is used. The drawings are made with a smaller sized die each time, until the desired diameter of wire is obtained. Cold drawing of steel and iron raises the elastic limit and ultimate strength of the metal and decreases its ductility. It is made ductile again by annealing, and is finished by giving it the proper temper consistent with the desired use.

The *Mining Journal* for 1896 gives the following values for the strength of wire.

	lb./in.^{2}		lb./in.^{2}
Iron wire	80,000	High-carbon steel wire . .	180,000
Bessemer steel wire . .	90,000	Crucible cast steel . . .	240,000
Mild open-hearth steel wire	130,000		

Piano wire varies in strength from 300,000 lb./in.^{2} to 400,000 lb./in.^{2}

261. Wire rope. Wire rope is made by twisting a number of steel or iron wires into a **strand**, and then twisting a number of these strands about one of the strands, or about a hemp, manila, jute, or cotton strand. The exact composition of the cable or wire rope will depend upon the service for which it is designed. The hemp core gives added pliability to the cable, and acts as a means of lubricating the strands and wires; this reduces the internal friction in the cable, and adds much to its life in case it is used where pliability is required, as in running over sheaves. Fig. 194 is an illustration of the cross section of a cable in which the separate strands each have a hemp core. Such a cable can be used where great pliability is required. Fig. 195 shows a cross section of a cable with

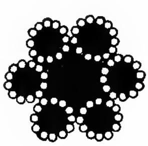


Fig. 194

a single hemp core at the center, and Fig. 196 shows a cross section of a cable in which the center is a wire strand similar to those used on the outside. A cable of the latter type can only be used where little bending is required, as in the case of suspension bridges. The strands are twisted about the central core either to the right or left. When twisted to the left the rope is designated as **left lay**, and when twisted to the right as **right lay**. The twist is long or short, depending upon the requirements of service. The shorter the twist the more flexible the rope, and the longer the twist the less flexible.

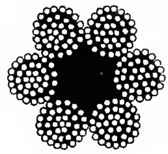


FIG. 195

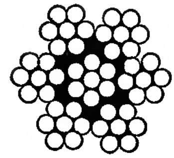


FIG. 196

262. Testing of rope wire and belting. These materials are usually tested in tension. This may be done in an ordinary testing machine, providing the proper means are used for holding the specimen. A type of wire-testing machine is shown in Fig. 197. One end of the wire is clamped to the movable head and the other to the stationary head, which is provided with a spring balance for registering the pull. Many other types of wire-testing machines are in use, some of them being arranged to make torsion tests. Many special machines are also made for testing rope and belting.

Since a wire rope is a **built-up** structure, made of twisted strands, it is not to be expected that it will exhibit such well-defined elastic properties as a single wire tested separately. This is due to the fact that as the tension is increased each strand, which was originally in the form of a helix of a certain pitch, becomes somewhat straightened and takes the form of a helix of a greater pitch. On account of the twisted condition of the wires in the strands, they do not all carry the same load, and therefore do not all reach their elastic limit at the same time. We find, consequently, upon testing a wire rope, that it has no well-defined elastic limit.

The individual wires of which the rope is made show a very high tensile strength and elastic limit, but exhibit no yield point, as the process of drawing seems to destroy the properties of the material that give the yield-point phenomena. The modulus of elasticity is not changed appreciably by the process of drawing.

Problem 315. A piece of steel music wire was tested in tension and the following data obtained. Draw the strain diagram, using loads in lb./in.^{2} as ordinates and unit elongations as abscissas, and find the elastic limit, the modulus of elasticity, and the modulus of elastic resilience. The wire was No. 25 gauge; diameter before test 0.0577 in., and sectional area 0.002615 sq. in. It was tested on a gauge length of 6 in. The sectional area at the point of fracture after test was 0.00132 sq. in. Compute the percentage of reduction of cross section.

TEST OF WIRE

Load lb.	Elongation in.	Load lb.	Elongation in.
100	.0058	660	.0627
200	.0146	680	.0661
300	.0223	700	.0698
400	.0316	720	.0752
500	.0415	740	.0791
520	.0450	760	.0852
540	.0463	780	.10936
560	.0489	800	.1039
580	.0512	838	Tensile strength, 320,460 lb./in.^{2}
600	.0546		
620	.0564		
640	.0591		

263. Strength of wire rope. The following report of tests of steel rope is taken from the *Watertown Arsenal Report,* 1889.

TENSION TESTS OF STEEL WIRE ROPE

Circumference in.	Number of Strands	Wires per Strand	Mean Diameter of Wires in.	Core	Sectional Area of Wire in.^{2}	Tensile Strength Total lb.	Tensile Strength Total lb./in.^{2}
1.5	6	18	.0321	Hemp	.0876	12,898	147,236
1.75	6	18	.0349	"	.1031	15,736	153,893
2	6	18	.0420	"	.1499	20,780	138,360
2.125	6	18	.0456	"	.1766	24,430	138,383
2.25	6	18	.0488	"	.2021	30,960	148,650
2.50	6	18	.0544	"	.2510	33,270	132,500
3	6	18	.0598	"	.3024	46,370	153,340
3.50	6	18	.0718	"	.4380	65,120	148,675
4.50	6	18	.0980	"	.8151	138,625	170,075

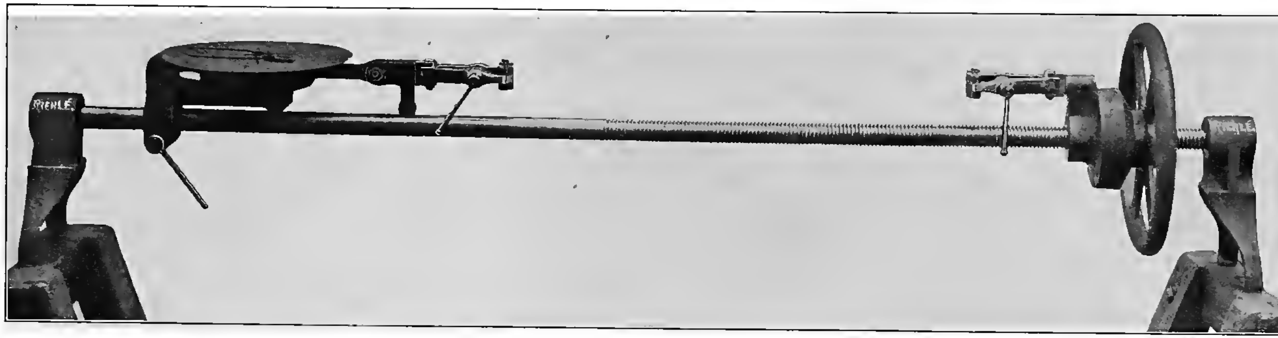


FIG. 197. — Machine for Testing Rope, Wire, and Cable

TEST OF INDIVIDUAL WIRES TAKEN FROM THE WIRE ROPE REPORTED ABOVE

Size of Rope in.	Diameter of Wire in.	Sectional Area in.²	Tensile Strength lb.	Tensile Strength lb./in.²
1.50	.0325	.00082	130	158,540
2.00	.0430	.00145	226	155,860
2.50	.0546	.00234	502	214,530
2.75	.0593	.00276	452	163,770
3.00	.0600	.00283	478	168,900
3.50	.0725	.00413	594	143,830
4.50	.9980	.00754	1390	184,350

In the above table the actual sectional area of the wire in the rope is given, and the tensile strength in lb./in.² has been computed by dividing the total load by this area. An examination of the table giving the strength of the individual wires shows that the intensity of stress is greater in the case of the individual wires than in the wire rope; that is to say, the structure of the rope causes the wires to lose some of their efficiency.

STRENGTH OF IRON WIRE ROPE AS GIVEN BY JOHN A. ROEBLING

(Rope composed of six strands and a hemp center, seven or twelve wires in each strand)

Diameter in.	Circumference in.	Approximate Breaking Strength lb.	Circumference in Inches of New Manila Rope of Equal Strength
1.75	5.50	88,000	11
1.625	5.00	72,000	10
1.50	4.75	64,000	9.5
1.375	4.25	52,000	8.5
1.25	4.00	46,000	8.0
1.125	3.50	36,000	6.5
1.000	3.00	26,000	5.75
.875	2.75	22,000	5.25
.750	2.25	14,600	4.75
.500	1.50	6,400	3.00
.375	1.125	3,600	2.25
.250	.75	1,620	1.50

The table at the bottom of page 359 gives the strength of iron and cast-steel wire rope as given by John A. Roebling's Sons. The size of a new manila rope of the same strength is also given for comparison.

STRENGTH OF WIRE ROPE MADE FROM CAST STEEL AS GIVEN BY JOHN A. ROEBLING

(Rope composed of six strands and a hemp center, seven or nineteen wires in each strand)

Diameter in.	Circumference in.	Approximate Breaking Strength lb.	Circumference in Inches of New Manila Rope of Equal Strength
1.25	4.00	106,000	13
2.125	3.50	82,000	11
1.00	3.00	62,000	9
.875	2.75	52,000	8.5
.750	2.25	35,200	7.0
.625	2.00	28,000	6.0
.500	2.50	16,200	4.75
.375	1.125	9,000	3.75

Problem 316. A wire cable of the following dimensions and composition was tested, and its maximum load found to be 5080 lb. Diameter of cable, 0.33 in.; six strands of eleven wires each; sectional area of wires, 0.0253 in.^{2} A test of the individual wires showed an average strength of 225,600 lb./in.^{2} Find the loss of strength due to the twisting of the wires to form the cable, assuming that all the wires have the average strength given above.

264. Strength of manila rope. The following table gives the strength of manila and sisal rope as computed from tests made by the Watertown Arsenal.* The load in lb./in.^{2} is given in each case. This has been computed by considering the cross section of the rope as the area of a circle of the same diameter. It will be seen from the table that the stress for the smaller ropes was 15,000 lb./in.^{2}, while for the larger ropes it was only about 7000 lb./in.^{2} This difference is due in part to the greater length of yarn used in the smaller rope. Manila rope has about two thirds the strength of good Russian hemp rope.† The United States Navy test allows 1700 lb./in.^{2} as the working strength of a 1.75-in. hemp rope.

* *Watertown Arsenal Report*, 1897.

† Thurston, *Materials of Construction.*

TESTS OF MANILA AND SISAL ROPE

Manila Rope

Size of Rope	Diameter in.	Sectional Area in.^{2}	Tensile Strength lb./in.^{2}	Tensile Strength Per Yarn lb.	Total Load lb.
6-thread	.27	.0567	13,360	126	750
9-thread	.30	.0750	14,180	118	1,064
12-thread	.38	.114	12,920	123	1,473
15-thread	.43	.153	14,250	145	2,180
1.25-in.	.49	.192	11,610	125	2,242
1.50-in.	.56	.259	11,970	148	3,100
1.625-in.	.61	.288	10,800	130	3,120
1.75-in.	.62	.299	11,500	128	3,455
2-in.	.74	.41	9,200	114	3,775
2.25-in.	.79	.478	12,900	148	6,207
2.25-in.	.78	.462	11,900	138	5,509
2.50-in.	.85	.557	12,470	136	6,947
2.75-in.	.96	.715	12,810	153	9,160
3-in.	1.00	.782	13,630	146	10,663
3-in.	.99	.746	13,750	151	10,260
3.25-in.	1.13	.970	12,470	144	12,093
3.50-in.	1.19	1.07	12,190	132	13,050
3.75-in.	1.29	1.27	11,990	134	15,227
3.75-in.	1.28	1.26	11,610	132	14,640
4-in.	1.39	1.46	10,080	117	14,723
4-in.	1.34	1.36	11,790	130	16,017
4.25-in.	1.41	1.51	9,890	113	14,943
4.50-in.	1.59	1.88	10,360	121	19,577
4.50-in.	1.61	1.99	10,480	134	20,873
4.75-in.	1.66	2.04	10,740	128	21,903
5-in.	1.76	2.35	9,940	118	23,360
6-in.	2.25	3.82	8,260	112	31,570
7-in.	2.52	4.86	9,400	125	45,647
8-in.	2.83	6.22	8,600	118	54,000
9-in.	3.35	8.37	7,500	108	62,717
10-in.	3.70	10.06	7,300	102	73,910

Sisal Rope

Size of Rope	Diameter in.	Sectional Area in.^{2}	Tensile Strength lb./in.^{2}	Tensile Strength Per Yarn lb.	Total Load lb.
6-thread	.27	.0567	7,700	72	432
9-thread	.33	.082	7,300	67	605
12-thread	.39	.126	7,500	79	944
1.25-in.	.45	.129	10,810	93	1397
1.50-in.	.56	.254	8,100	99	2067
1.75-in.	.63	.302	7,600	96	2315
2-in.	.70	.395	7,200	97	2925
2.25-in.	.81	.416	9,500	94	3966
2.75-in.	.95	.691	8,300	101	5733
3-in.	1.01	.780	7,500	104	5917
3.50-in.	1.22	1.128	7,200	102	8230

265. Strength of leather and rubber belting. Leather belts are made from tanned oxhide. That portion of the hide that originally covered the back gives the best leather for this purpose. The "flesh side," or side originally next to the animal, wears better when placed in contact with the pulley, while the outside gives the greater adhesion when placed in contact with the pulley.

Single belts are made from one thickness of leather, the desired length being obtained by cementing or splicing the short lengths cut from the hide. **Double belts** are made by cementing two thicknesses of the leather together. The strength of good leather varies from 600 to 700 lb. per inch of width, and from one half to two thirds as much when spliced. The following table gives the strength of cemented belt laps as determined by the Watertown Arsenal.* A complete series of tests on belt lacings is also reported in the same volume, and the student is referred to this report for the results. The allowable stress on a single belt is from 250 to 300 lb. per inch of width.

TESTS OF LEATHER BELTING

Description †	Dimensions in.			Sectional Area in.²	Tensile Strength	
	Length	Width	Thickness		lb./in.²	Pounds per Inch of Width
2-in., single	60.00	1.98	.20	.396	5045	1091
6-in., single	60.20	6.07	.22	1.34	2537	560
6-in., single (*w*)	60.11	6.08	.24	1.46	2119	533
12-in., single	60.11	12.05	.18	2.17	3917	705
4-in., double	59.55	3.98	.33	1.31	4931	1623
6-in., double	60.18	5.91	.47	2.78	4309	2027
6-in., double (*w*)	59.93	6.00	.40	2.40	5166	2066
12-in., double	59.90	11.90	.39	4.64	4090	1595
12-in., double (*w*)	60.06	11.93	.36	4.29	4424	1591
24-in., double (*w*)	60.00	23.90	.47	11.23	2760	1297
30-in., double	59.90	29.95	.43	12.88	2717	1169

* *Watertown Arsenal Report*, 1893.
† The letter *w* in the table stands for *waterproofed*.

TESTS OF RUBBER BELTING

Description	Dimensions in.			Sectional Area	Tensile Strength	
	Length	Width	Thickness	in.^{2}	lb./in.^{2}	Pound per Inch of Width
2-in., 4-ply	60.17	2.02	.26	.525	3276	851
6-in., 4-ply	60.17	6.08	.26	1.58	3227	839
6-in., 4-ply	60.12	6.13	.26	1.59	3773	979
6-in., 4-ply	60.17	6.05	.26	1.57	2739	711
12-in., 4-ply	60.02	12.08	.27	3.26	3037	819
12-in., 4-ply	60.14	12.24	.26	3.18	2987	776
2-in., 6-ply	60.17	2.14	.36	.770	3104	1116
6-in., 6-ply	59.98	6.26	.37	2.32	2737	1014
6-in., 6-ply	60.08	6.27	.36	2.26	3770	1358
12-in., 6-ply	60.15	12.04	.36	4.33	3436	1236
12-in., 6-ply	60.17	12.16	.34	4.13	3862	1311
24-in., 6-ply	60.13	24.11	.41	9.89	2381	977
30-in., 6-ply	60.04	30.18	.40	12.07	2808	1123

ANSWERS TO PROBLEMS

1. 17.7 lb./in.^{2}
2. 3.1 lb./in.^{2}
3. $s = .0018$.
4. 5.4 in.
5. .0000104.
6. $s = .002122$, 12.73 in.
7. 16,500,000 lb./in.^{2}, approximately.
8. .0055 in., approximately. $s = .00002546$.
9. 31,024 lb.
10. .000307 in.
11. .00095 in.
12. 1005 lb.
13. .24 in. square.
14. $\frac{1}{8}$ in.
15. 99 tons.
16. 20.0086 ft.
17. 1890 lb./in.^{2}
18. .16 in.
19. 13,320 lb./in.^{2}
20. 1.7, approximately.
21. $18\frac{1}{2}$.
22. $1\frac{1}{8}$ in. iron wire rope.
40. 320 lb./in.^{2}, 19° 19.8′, 109° 19.8′.
41. $p' = 2000$ lb./in.^{2}, $q' = 3460$ lb./in.^{2}, $q'_{\max} = 4000$ lb./in.^{2}
42. 10,000 lb./in.^{2}
43. 58,435 lb.
44. 23,868 lb. direct stress, 77,748 lb. shear.
45. 5656 lb./in.^{2}
46. With a factor of safety of 5, $d = 1.21$ in.
47. $m = 5$.
48. $p_e = 556$ lb./in.^{2} for $m = 3\frac{1}{3}$.
49. $p_e = 12{,}646$ lb./in.^{2} for $m = 3\frac{1}{3}$.
57. $\dfrac{bh^3}{36}$.
58. $\dfrac{\pi d^4}{64}$.
60. 35,350 ft. lb.
61. 942,500 ft. lb.
62. $M_1 : M_2 = 261.3 : 149.3$.
63. $\dfrac{bh^2}{6}$, $\dfrac{bh^2}{24}$, $\dfrac{\pi d^3}{32}$.
64. $S_1 : S_2 = 5 : 2$.
66. $\dfrac{bh}{48}(3h^2 + b^2)$.
67. $\frac{1}{12}(bh^3 - b'h'^3)$.
69. $I_p = \dfrac{\pi d^4}{32}$, $t_p = \dfrac{d}{2\sqrt{2}}$.
76. Zero at center, 1500 lb. at ends.
77. 4250 lb. and 4750 lb. at ends, 2750 lb. and 1750 lb. between loads.
78. At center; 1837.5 ft. lb.
79. 408 lb./in.^{2}
80. 14,603 lb./in.^{2}
84. $S = 66.46$ in.^{3}
85. In the ratio 1 : 96.

120. $y = \dfrac{P}{6EI}(2l^3 - 3l^2x + x^3)$, $D = \dfrac{Pl^3}{3EI}$.

122. $y = \dfrac{wx}{24EI}(l^3 - 2lx^2 + x^3)$, $D = \dfrac{5wl^4}{384EI}$.

123. $y = \dfrac{w}{24EI}(x^4 - 4l^3x + 3l^4)$, $D = \dfrac{wl^4}{8EI}$.

124. $D = .7$ in.
125. $D = .67$ in.
127. $D = \dfrac{Pd^3(l-d)^3}{3EIl^3}$.
128. $D = \dfrac{Pl^3}{192EI}$.
129. $D = .061$ in. for $E_c = 2{,}000{,}000$ lb./in.^{2}
134. $M_1 = M_6 = 0$, $M_2 = M_5 = -\frac{2}{19}wl^2$, $M_3 = M_4 = -\frac{3}{38}wl^2$, $F_1 = F_6 = \frac{15}{38}wl$, $F_2 = F_5 = \frac{43}{38}wl$, $F_3 = F_4 = \frac{37}{38}wl$.
136. $W = 39.7$ in. lb.
138. $h = 7.86$ in.
170. 350 tons.
171. $9\frac{1}{4}$ in. square.
172. 5.82 in.
173. $6\frac{1}{8}$ in. wide for angles $\frac{1}{2}$ in. thick.
174. Rankine 616 tons, Johnson 627 tons.
175. Rankine 268 tons, Johnson 267 tons.
176. Assume various lengths for the column.

177. 127 tons.

178. 15 +

179. $2\frac{3}{8}$ in. square.

190. $G = 11{,}490{,}000$ lb./in.^{2}

191. $M = 43.24$ in. lb.

192. $d = 4.465$ in.

193. $d = 3.684$ in.

194. Internal diameter = 5.63 in.; solid : hollow = 3 : 1.

195. 4484.

196. $p_e = 23{,}500$ lb./in.^{2}

197. If weight of shaft is neglected, $q = 139$ lb./in.^{2}, $H = 2\frac{1}{4}$.

198. $d = 7.114$ in.

199. $\theta = 32^\circ\, 28'$.

201. Angle of twist per unit of length is $\theta_1 = 0^\circ\, 1'\, 33.8''$.

202. $\theta_1 = 0^\circ\, 0'\, 59.3''$.

203. $q_{\max} = 22{,}240$ lb./in.^{2}, $D = 6.36$ in., $W = 158.965$ in. lb.

221. 2940 lb./in.^{2}

222. 375 lb./in.^{2}, assuming 10 for the factor of safety.

223. Bottom .13 in.; side .31 in.

224. 6528 lb./in.^{2}

225. $\frac{4}{9}$ in.

226. 685 lb./in.^{2}

227. $68\frac{1}{2}$.

228. $\frac{1}{4}$ in.

229. .13 in.

230. 5911 lb./in.^{2}

231. 15,880 ft.

232. 79.4.

233. 12,187 lb./in.^{2}

249. 1.2 in.

250. 2344 lb./in.^{2}

252. 139 lb./in.^{2}

253. .28 in.

254. 11.78 lb./in.^{2}

255. Assuming $E_s : E_c = 15 : 1$, $I' = 2350$ in.^{4}, $e' = 2.266$ in., $p = 450$ lb./in.^{2}

270. $p_{\max} = 3733$ lb./in.^{2}, factor of safety 13, $d = .0245$ in.

272. $p_{\max} = 192.6$ lb./in.^{2}

273. $d = .0002$ in., $M = 1.529$ in. lb.

289. $R = 300$ tons, by (104); $R = 327$ tons, by (105).

291. 3 in.

293. Weyrauch, 4042 lb./ft.; Rankine, 4116 lb./ft.

294. 4242 lb./ft.

295. 13,890 lb./ft.

300. 2250 lb./in.^{2}, 3091 lb./in.^{2}

302. 476 lb./in.^{2}

303. 450 lb./in.^{2}

310. (*a*) 12,870 lb./in.^{2}, 13,059 lb./in.^{2}; (*b*) 2659 lb./in.^{2}, 8372 lb./in.^{2}; (*c*) 1908 lb./in.^{2}, 5538 lb./in.^{2}

312. 752 lb./in.^{2}

INDEX

(The numbers refer to pages.)

ANNOUNCEMENTS

APPLICATIONS OF THE CALCULUS TO MECHANICS

By E. R. HEDRICK, Professor of Mathematics in the University of Missouri,
and O. D. KELLOGG, Assistant Professor of Mathematics
in the University of Missouri

8vo, cloth, 116 pages, with diagrams, $1.25

THIS book presents a completed summary of those parts of mechanics which occur as applications of the calculus. Although intended primarily as a supplement to the usual standard course in calculus, it may be used independently as a text for a short course on the mathematical side of mechanics, if the time allotted to the former study is not sufficient to include this work.

As a review it fastens in the student's mind the notions of mechanics previously gained. It aims also to present these topics in a new light, as articulated portions of one general theory, and thus to make mechanics seem an integral subject.

As a preparation for more extended courses in mechanics, or indeed for courses dealing with any applications of the calculus, the material presented is valuable in showing concretely how the theoretical ideas of this subject are used in specific practical applications.

The course outlined in the book is the result of a number of years' experience in presenting this material to classes in the calculus, both at the University of Missouri and elsewhere. The text itself is a modification of a similar text written by Professor Hedrick and published in mimeograph for the use of students at the Sheffield Scientific School.